Ph.D. COMPREHENSIVE EXAM - TOPOLOGY - JUNE 1995

1. (12 points) Let X be a Hausdorff space. Suppose $A \subseteq X$ is a retract of X. Prove that A is closed in X.

2. (12 points) Suppose $f : X \to Y$ is continuous and onto, X is compact, and Y is Hausdorff. Prove that f is a quotient map.

3. (12 points) Prove that every metrizable Lindelöf space X is second countable. Hint: Let $\mathcal{U}_n$ be a covering of X by open $1/n$-balls.

4. (12 points)

   (a) State the Urysohn Lemma.

   (b) Prove that every connected normal space having at least two points is uncountable.

5. (12 points) Let M_g denote the connected sum of g tori; let N_g denote the connected sum of g projective planes. Classify each of the following surfaces as an M_g or N_g, specifying g.

   (a) The connected sum of a torus and a Klein bottle.

   (b) The compact surface S with $\chi(S) = -7$.

   (c) The compact, orientable surface S with $\chi(S) = -8$.

6. (12 points) Let T be a torus and P a projective plane. Give presentations for the fundamental groups of the following spaces. (Don't just say what the groups are algebraically; give *presentations* for them.)

   (a) $T \times P$

   (b) $T \# P$, where "#" denotes connected sum.

   (c) $T \vee P$, where "$\vee$" denotes the gluing of the two spaces together at a single point.

7. (12 points) Define $p : \mathbf{R}^3 \to \mathbf{R}^2 \times \{0\}$ by $p(x, y, z) = (x, y, 0)$, i.e. p is just projection onto the x-y plane. Let $D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ be the unit disk in the x-y plane. Suppose E is a disk embedded in $\mathbf{R}^3$ (perhaps in a very peculiar manner) such that $\partial E = \partial D = \{(x, y, 0) \mid x^2 + y^2 = 1\}$, i.e. the unit circle in the x-y plane. Prove that $(0, 0, 0) \in p(E)$. Hint: Consider $\mathbf{R}^3$ minus the z axis. Remark: With slightly more work one can show that $D \subseteq p(E)$.

8. (12 points) Let $p : \widetilde{X} \to X$ be a covering space. $\widetilde{X}$ and X are assumed to be path connected and locally path connected. Let $x, y \in X$. Prove that there is a bijection $\varphi : p^{-1}(x) \to p^{-1}(y)$.

9. (4 points) State the Seifert-Van Kampen Theorem for the case of a space X which is the union of two subspaces U and V. Be sure to state all the hypotheses.