Master's Comprehensive Exam – Advanced Calculus – May 1996

Do 9 of these 10 problems

1. (a) Define Riemann integrability on $[a, b]$ for a function $f : [a, b] \to R$.

   (b) Consider $f : [0, 1] \to R$ defined by

$$f(x) = \left\{ \begin{array}{ll} 1, & for\ x = 0 \\ 0, & for\ x \neq 0 \end{array} \right\}$$

   Using only the definition, show that f is Riemann integrable on $[0, 1]$.

2. Suppose that $f : R \to R$ is differentiable, $f'(x) \geq 3$, and $f(0) = 0$. Show that $f(10) \geq 30$.

3. Consider the function $f : R^2 \to R$ defined by

$$f(x, y) = 2x^4 + y^4 + x^2 - 2y^2.$$

   Find the critical or stationary points and discuss the nature of each one.

4. (a) Define what it means for a sequence p_n of points to be a Cauchy sequence.

   (b) Show that any convergent sequence is a Cauchy sequence.

5. (a) Define compactness for a subset S of the real line R.

   (b) Show that any compact set is bounded.

6. For the relationship $x^2 + y^2 - x^3 = 0$ explicitly find a solution of the form $y = f(x)$ valid near the point $(5, -10)$.

7. Does the following series converge pointwise on the reals $\mathbf{R}$? Does it converge uniformly on $\mathbf{R}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$$

8. Suppose that $a_n \geq 0$. Show that if $\sum_{n=1}^{\infty} a_n$ converges then so does $\sum_{n=1}^{\infty} a_n^2$.

9. (a) Define bounded variation on $[a, b]$ for a function $f : [a, b] \to R$.

   (b) Suppose that $f : [a, b] \to R$ is continuous and f' is bounded. Show that f is of bounded variation.

10. (a) Define what it means for a point x to be a boundary point of a set $S \subset R^n$.

    (b) Show that if x is a boundary point of $S \subset R^n$ then there is a sequence $\{s_n\}$ of points from S such that $\lim_{x \to \infty} s_n = x$.

    (c) Is it true or false that if x is a boundary point of S then it is also possible to find a sequence $\{s_n\}$ with $s_n \notin S$ such that $\lim_{x \to \infty} s_n = x$?