Ph.D. COMPREHENSIVE EXAM: TOPOLOGY

May 1996

1. Let X and Y be topological spaces. Give $X \times Y$ the product topology. Let $A \subseteq X$ and $B \subseteq Y$. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

2. Recall that a topological space X is locally path connected if for every $x \in X$ and open neighborhood U of x there is a path connected open neighborhood V of x such that $V \subseteq U$. Prove that every connected, locally path connected topological space is path connected.

3. Prove that every compact Hausdorff space is normal.

4. Prove that every separable metric space satisfies the second axiom of countability.

5. For each of the following pairs X, Y of topological spaces state whether or not X and Y are homeomorphic. You do not have to prove your answers. All these spaces have the standard subspace topology as subsets of some Euclidean space.

   (a) $X = [0, 2\pi)$, $Y = S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$

   (b) $X = (-1, 1)$, $Y = \mathbf{R}$

   (c) $X = \mathbf{R}$, $Y = \mathbf{R}^2$

   (d) $X = \mathbf{Z}_+$ (the set of positive integers), $Y = \{\frac{1}{n} \mid n \in \mathbf{Z}_+\}$

6. Let M_g denote the connected sum of g tori; let N_g denote the connected sum of g projective planes. Classify each of the following surfaces as an M_g or N_g, specifying g.

   (a) The orientable surface with Euler characteristic -6.

   (b) The connected sum of a torus and a Klein bottle.

7. Write down presentations for the fundamental groups of the following spaces.

   (a) The connected sum of two tori.

   (b) A space whose universal covering space is 2-sheeted.

8. Let $p : \widetilde{X} \to X$ be a covering map, where $\widetilde{X}$ and X are path-connected and locally path connected. Suppose $p(\tilde{x}_0) = x_0$. Prove that $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is one-to-one.

9. Give examples of topological spaces with the following fundamental groups.

   (a) $\mathbf{Z} \times \mathbf{Z}$

   (b) $\mathbf{Z} * \mathbf{Z}$

   (c) $\mathbf{Z} * (\mathbf{Z} \times \mathbf{Z})$

   (d) $(\mathbf{Z} * \mathbf{Z}) \times \mathbf{Z}$

10. Let $a \in A \subseteq X$. Let $i : A \to X$ be the inclusion map. Suppose $r : X \to A$ is a retraction.

    (a) What can be said about $r_* : \pi_1(X, a) \to \pi_1(A, a)$?

    (b) What can be said about r_* if r is a (strong) deformation retraction?

    Prove your answers.