MASTER'S COMPREHENSIVE EXAM—ADVANCED CALCULUS
JUNE, 1997
All problems have equal value

1. Let a be a positive integer and define $f(x) = \begin{cases} x^a \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$
   (a) For what values of a is f differentiable at 0 and what is $f'(0)$?
   (b) For what values of a is f' continuous at 0?

2. (a) Let $f : \mathbf{R}^2 \to \mathbf{R}$. Define the phrase "$f$ is differentiable at the point $(a, b) \in \mathbf{R}^2$."
   (b) Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ are differentiable at (a, b). Define $F : \mathbf{R}^2 \to \mathbf{R}$ by the formula
   $$F(x, y) = f(x, y) + g(x, y).$$
   Prove directly from the definition that F is differentiable at (a, b).

3. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function satisfying $D_1 f(0, 0) = D_2 f(0, 0) = 10$. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by the formula
   $$g(x, y, z) = f((y - x)z, (z - x)y).$$
   Calculate $D_i g(1, 1, 1)$ for $i = 1, 2, 3$.

4. Find all real values of x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges, where
   $$a_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ is odd} \\ \left(\dfrac{1}{2}\right)^n & \text{if } n \text{ is even.} \end{cases}$$

5. Consider the sequence of functions $f_n(x) = (x/n)e^{-x/n}$.
   (a) Show that this sequence converges pointwise for all $x \in \mathbf{R}$.
   (b) Show that this sequence converges uniformly on $[0, C]$ for all $C > 0$.
   (c) Does this sequence converge uniformly on $[0, +\infty)$?

6. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$. Suppose $(x_0, y_0, u_0) \in \mathbf{R}^3$ is a solution of the system
   $$\begin{aligned} f(x) + f(y) &= u \\ g(x) + g(y) &= u^2. \end{aligned}$$
   State conditions on f and g that guarantee that this system has solutions $x = p(u)$ and $y = q(u)$ defined in a neighborhood of u_0 and satisfying $p(u_0) = x_0$ and $q(u_0) = y_0$.

7. Let $f : [a, b] \to \mathbf{R}$.
   (a) Define the phrase "f is Riemann integrable on $[a, b]$."
   (b) Prove directly from the definition that the function $f : [0, 1] \to \mathbf{R}$ defined by
   $$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$
   is not Riemann integrable on $[0, 1]$.