Master's Comprehensive Exam
ADVANCED LINEAR ALGEBRA
June 1997

1. (30 pts.) Let
$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 2 & -6 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$
   (a) Find the rank of A.
   (b) Give a basis for the kernel $\ker(A)$.
   (c) Give a basis for the row space $\mathrm{Row}(A)$.
   (d) Give a basis for the column space $\mathrm{Col}(A)$.
   (e) Give the relationship between $\ker(A)$ and $\mathrm{Row}(A)$.

2. (20 pts.) A matrix A has characteristic polynomial
$$c_A(x) = (x-3)^4(x+2)^2.$$
   (a) State all possible minimal polynomials for A, and give examples of matrices A with characteristic polynomial $c_A(x)$ and with each possible minimal polynomial.
   (b) Give an example of two matrices A and B which have the same characteristic polynomial and the same minimal polynomial yet are *not* similar. Explain why the matrices are not similar.

3. (25 pts.) Consider a vector space V over $\mathbf{R}$ with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
   (a) Define what we mean by a *bilinear form* b on V.
   (b) Give the matrix $A = (a_{ij})$ for the bilinear form b on V with respect to the basis $\mathcal{B}$.
   (c) Suppose that $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a new choice of basis for V and that P is the corresponding change of basis matrix. Describe the matrix for b with respect to $\mathcal{C}$ in terms of the old matrix A and the change of basis matrix P.
   (d) Consider the bilinear form b on $\mathbf{R}^3$ given by
$$b(x,y) = x_1y_1 + 2x_2y_2 + 4x_3y_3 - x_1y_2 - x_2y_1 + 2x_2y_3 + 2x_3y_2.$$
   Find a basis of $\mathbf{R}^3$ with respect to which the matrix of b is diagonal, and give the matrix of b with respect to your choice of basis.

4. (25 pts.) Let V be a vector space over a field $\mathbf{F}$ with inner product $\langle v, w\rangle$. Prove Bessel's inequality that whenever $v_1, v_2, \ldots, v_n \in V$ are nonzero and orthogonal and $v \in V$, then

$$\sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\langle v_i, v_i\rangle} \leq \langle v, v\rangle.$$

(Hint: Write $v = w_1 + w_2$ with $w_1 \in \operatorname{span}\{v_1, \ldots, v_n\}$ and $w_2 \in \operatorname{span}\{v_1, \ldots, v_n\}^{\perp}$.)