Masters Algebra Examination
June 1997

1. A normal subgroup $H \neq \{e\}$ of G is called a minimal normal subgroup of G iff H has no proper subgroups which are normal in G. Prove: If H and N are distinct minimal subgroups of G then HN/N is isomorphic to H.

2. If f is in S_n and $(i_1,\ldots,i_r)$ is a cycle in S_n then find the cycle notation for $f(i_1,\ldots,i_r)f^{-1}$. Prove the result by showing the calculation. Here $fg(x)=f(g(x))$.

3. Prove: Every finitely generated vector space $(\neq 0)$ has a basis by showing that any generating set contains a basis.

4. Let F be a field, V and W vector spaces over F and $\alpha : V \to W$ be an F homomorphism. Prove: α is 1-1 iff for f and g in $\mathrm{Hom}_F(F,V)$, $\alpha \circ f = \alpha \circ g$ implies f=g.

5. Let R be a principal ideal domain and let $\{a_1,a_2,\ldots,a_n\}$ be a subset of R, with n>0 and not all the a_i=0. Prove that this set has a gcd, and any gcd of it is of the form $\Sigma r_i a_i$ for some r_i in R.

6. If F is a field the prove that F[x] is a principal ideal domain.

7. Let F be a field and p(x) be an irreducible polynomial in F[x]. Show how to construct a field K with the following properties:
   a) p(x) has a root in K.
   b) There is a 1-1 field homomorphism from F to K.
Be sure to show that these properties are satisfied by K.

8. The polynomial $f(x)= x^3 - 5$ has roots $5^{\frac{1}{3}}, \omega 5^{\frac{1}{3}}, \omega^2 5^{\frac{1}{3}}$ over Q, the field of rational numbers. $(\omega = \frac{-1+i3^{\frac{1}{2}}}{2})$ Find the Galois group of x^3-5 over Q. Make explicit the correspondence between subgroups of the Galios group and intermediate extensions between Q and the splitting field of f(x).