Ph.D. EXAM: TOPOLOGY
June 1997

1. Prove that every path connected space is connected.

2. Let A and B be compact subsets of a Hausdorff space X. Prove that $A \cap B$ is compact. Hint: Do not use the definitions of "compact" and "Hausdorff." Warning: The statement is not in general true for non-Hausdorff spaces.

3. Prove that if X is a metric space which has a countable dense subset, then X satisfies the second axiom of countability.

4. Let G be a group of self-homeomorphisms of a space X. Define an equivalence relation on X by setting $x \sim x'$ if and only if there exists a $g \in G$ such that $g(x) = x'$. Let Y be the set of equivalence classes, and let $p : X \to Y$ be the function which takes x to its equivalence class. Give Y the quotient topology induced by p.

   (a) Prove that p is an open map.

   (b) Prove that if G is finite, then p is a closed map.

   Hint: For $A \subseteq X$ one has $p^{-1}(p(A)) = \bigcup_{g \in G} g(A)$.

5. Prove that every retract A of a Hausdorff space X is closed in X. Hint: Show that $X - A$ is open.

6. Let M_g denote the connected sum of g tori; let N_g denote the connected sum of g projective planes. Classify each of the following as an M_g or N_g, specifying g.

   (a) The orientable surface (without boundary) having Euler characteristic -14.

   (b) The connected sum of a torus and a Klein bottle.

7. Let X be a path connected space. Let $x, y \in X$. If γ is a path class from x to y, then denote by $\widehat{\gamma} : \pi_1(X, x) \to \pi_1(X, y)$ the change of basepoint isomorphism given by $\widehat{\gamma}(\alpha) = \gamma^{-1} \cdot \alpha \cdot \gamma$. Prove that $\widehat{\gamma} = \widehat{\delta}$ for every pair of path classes γ and δ from x to y if and only if $\pi_1(X, x)$ is abelian.

8. Let $X \vee Y$ denote the space obtained by joining the spaces X and Y together at a single point. Write down the fundamental groups of the following spaces.

   (a) $(S^1 \vee S^1) \times S^1$

   (b) $S^1 \vee (S^1 \times S^1)$

9. Let $p : \widetilde{X} \to X$ be a covering map, where $\widetilde{X}$ and X are path connected and locally path connected. Let $\tilde{x} \in \widetilde{X}$ and $x = p(\tilde{x})$. Prove that $p : \widetilde{X} \to X$ is one to one if and only if $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$ is onto.

10. Prove that $\mathbf{R}^2$ and $\mathbf{R}^3$ are not homeomorphic.