Comprehensive Examination
Advanced Calculus
June 1998

This exam consists of seven problems to be completed in two hours. A passing score will consist of five problems completed correctly. Please show all work.

1. Let $\{a_n\}$, $\{b_n\}$ be bounded sequences of real numbers.
   (a) Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
   (b) Either prove that equality holds in part (a) or give an example to show that strict inequality can occur in part (a).
2. Let X be a metric space.
   (a) Define the term *compact* in terms of open covers.
   (b) Using the definition in the previous part prove that if C is a compact subset of X and $S \subseteq C$ is infinite then S has a limit point in C.
3. Let $f : [0, \infty) \to \mathbf{R}$ be a continuous function that is differentiable on $(0, \infty)$. Suppose that $f(0) > 0$ and $f'(x) < 0$ for all $x > 0$. Prove that the equation $f(x) = x$ has one and only one solution.
4. Determine the radius of convergence and the convergence at the endpoints of the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_n = (\sin \frac{n\pi}{3})^n$.
5. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 - x^2}$ converges uniformly on any closed interval $[a, b]$ that does not contain any of the points $\pm 1, \pm 2, \ldots$.
6. Let $f : \mathbf{R}^n \to \mathbf{R}$.
   (a) Define the phrase "f is differentiable at $(0, 0, \ldots, 0)$."
   (b) Prove directly from the definition that $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x^2 + y$ is differentiable at $(0, 0)$.
7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and define $g : S \to \mathbb{R}$ by
   $$g(x, y, z) = x^3 f\left(\frac{y}{x}, \frac{z}{x}\right),$$
   where $S = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0\}$. Show that
   $$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + z\frac{\partial g}{\partial z} = 3g.$$