Masters Algebra Examination
Oklahoma State University
Department of Mathematics
June, 1998

1. Exactly two of the following three groups are isomorphic.

$$\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \qquad \mathbb{Z}/36\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \qquad \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$$

Identify which two are isomorphic and prove that those two are isomorphic, and prove that they are not isomorphic to the remaining group.

2. (a) Define Sylow subgroups of a finite group, and state the Sylow theorems as completely as possible.
   (b) Using these theorems, prove that there are no simple groups of order 56.

3. Let V be a vector space over a field F and let $W \subset V$ be a subspace. Show that

$$\dim(V) = \dim(W) + \dim(V/W).$$

4. (a) Show that $\mathbb{Q}[x]$ is a principal ideal domain.
   (b) Let I be the ideal of $\mathbb{Q}[x]$ generated by the polynomials $q_1(x) = 4x^3 - 8x^2 + 4x$ and $q_2(x) = 2x^3 - 6x^2 + 4x$. Find a polynomial $p(x)$ such that $I = (p(x))$.
   (c) Show that $\mathbb{Q}[x]/I$ is not an integral domain. Can I be a prime ideal?

5. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 3, not necessarily irreducible.
   (a) What are the possible Galois groups of $f(x)$ over $\mathbb{Q}$?
   (b) Give examples of cubic polynomials $f(x)$ with each possible Galois group from (a).
   (c) For each polynomial in (b), give the splitting field $E/\mathbb{Q}$ of $f(x)$ and the action of the Galois group on E as field automorphisms.