Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
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1. Let p, q and r be prime numbers with $r > pq$. Prove that any group G of order pqr is solvable.

2. Let M be a submodule of the finitely generated module N over the principal ideal domain R. Prove that M is finitely generated.

3. Let R be a ring. Show that an R-module M is isomorphic to $A_1 \bigoplus A_2$ if and only if there are R-homomorphisms $f_i : A_i \to M$, $i = 1, 2$, such that for any module B and R-homomorphisms $g_i : A_i \to B$, $i = 1, 2$, there is a unique R-homomorphism $h : M \to B$ so that $g_i = h \circ f_i$.

4. (a) For each prime p, show there exists a field $\mathbb{F}_p$ with p elements.
   (b) For each $n \geq 2$ show that there exists a field $\mathbb{F}_q$ with $q = p^n$ elements.
   (c) If p_1 and p_2 are distinct primes, does there exist a field with $p_1 p_2$ elements? Prove your answer.

5. Let $L = \mathbb{Q}(\zeta_{13})$ with ζ_{13} a primitive 13th root of unity.
   (a) Prove that there exists a unique subfield $K \subset L$ with $[K : \mathbb{Q}] = 4$.
   (b) Prove that K is Galois over $\mathbb{Q}$ and that the Galois group of K over $\mathbb{Q}$ is cyclic.
   (c) Let $\alpha \in K$ be an element with $K = \mathbb{Q}(\alpha)$. Prove that $\mathbb{Q}(\alpha^3) \neq \mathbb{Q}$.
   (d) Prove that $\mathbb{Q}(\alpha^3) = K$.

6. Let G be a finite group. G is **metabelian** if there exists an abelian normal subgroup H of G such that G/H is abelian.
   (a) Prove that if G is metabelian and K is a homomorphic image of G then K is metabelian.
   (b) Prove that if A is a subgroup of a metabelian group then A is metabelian.