PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—June 1998

Notation: $B = \{z \in \mathbf{C}\colon |z| < 1\}$, the open unit disc

1. Suppose that f, g, and $\overline{f}g$ are all analytic in B. Show that either f is constant or g is identically 0.

2. (a) Suppose that f is analytic in B. Show that $f = F'$ for some F analytic in B.

   (b) Let $B' = B \setminus \{0\} = \{z \in \mathbf{C}\colon 0 < |z| < 1\}$. Define $f(z) = \frac{1}{z}$. Show that there does *not* exist a function F analytic in B' with $F' = f$.

   (c) Explain the difference between B and B' here, by filling in the blank in the following theorem: Suppose G is a connected open subset of the plane. Every analytic function in G is the derivative of an analytic function if and only if G is ____________________. (No proof required here, just a correct statement).

3. (a) Suppose that u is a (real-valued) harmonic function in B. Show that u is the real part of some function f analytic in B.

   (b) Define $B' = B \setminus \{0\}$ as in the previous problem. Let $u(z) = \log(|z|)$. Show that u is *not* the real part of any function analytic in B'.

   (One way to do part (a) would be to show somehow that the function $u_x - iu_y$ was analytic, and then use that fact somehow.)

4. Suppose that f is an entire function with period 1 and period i. (That is, f is analytic in the entire plane and $f(z+1) = f(z+i) = f(z)$ for all z.) Show that f is constant.

5. Suppose that f is analytic and one-to-one in a neighborhood of the closed disc $\overline{B}$. Let $\gamma = \partial B$, oriented counterclockwise. Show that the inverse function f^{-1} is given by the formula
$$f^{-1}(\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-\alpha}\,dz$$
   for any $\alpha \in f(B)$.

6. Let $\mathcal{F}$ denote the set of all f analytic on B such that
$$\iint_B |f(x+iy)|^2\,dxdy \le 1.$$
   Show that $\mathcal{F}$ is a normal family (that is, $\mathcal{F}$ is uniformly bounded on compact subsets of B).
   Hint: Fix $p \in B$. Show first that
$$|f(p)|^2 \le \frac{1}{2\pi}\int_0^{2\pi} |f(p+re^{it})|^2\,dt$$
   for $0 < r < 1 - |p|$. It follows from this (how?) that
$$|f(p)|^2 \le \frac{1}{\pi(1-|p|)^2}\iint_{|x+iy-p|<1-|p|} |f(x+iy)|^2\,dxdy,$$
   and the result follows.