Comprehensive Examination
Real Analysis
June 1998

A passing mark will be given if 5 problems are completely solved.

1. Let $X = [0,1] = Y$, $\mathcal{A} = \{A : A \subset X\}$
   and $\mathcal{B} = \{B : B \subset X, B \text{ is countable or } X \setminus B \text{ is countable}\}$.
   (a) Define *σ-algebra* and show that $\mathcal{B}$ is a σ-algebra.
   (b) Show that there is a non-measurable map from $(X, \mathcal{B})$ to $(Y, \mathcal{A})$ and that every map from $(Y, \mathcal{A})$ to $(X, \mathcal{B})$ is measurable.
2. Suppose that f is a monotone non-decreasing real-valued function on $[0, 1]$.
   (a) Show that f has at most countably many points of discontinuity.
   (b) Prove or disprove: There is a sequence of absolutely continuous functions (f_n) such that $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [0, 1]$.
3. Suppose that $f : X \to R$ is measurable with respect to the measure space $(X, \mathcal{M}, \mu)$ and, for any $x \in X$, $0 \leq f(x) \leq 1$. Let $E = f^{-1}(1)$. Also, suppose that $g \in L^1(\mu)$.
   Prove that
   $$\lim_{n\to\infty} \int f^n \, g \, d\mu = \int_E g \, d\mu.$$
4. Let
   $$f(x) = \begin{cases} x & \text{if } 0 \leq x < .5 \\ 2x & \text{if } .5 \leq x \leq 1 \end{cases}$$
   (a) What is the Lebesgue-Stieltjes measure ν on [0,1] induced by f? (What is an explicit formula for calculating $\nu(A)$ for all Borel sets A?)
   (b) State the Lebesgue Decomposition Theorem.
   (c) Determine the Lebesgue Decomposition of ν with respect to Lebesgue measure on $[0, 1]$.
5. Suppose that $1 \leq r < p < \infty$.
   (a) Show that $L_p[0, 1] \subset L_r[0, 1]$.
   (b) Prove or disprove: $L_p[0, \infty) \subset L_r[0, \infty)$.
6. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of differentiable functions in $C([0, 1])$ such that for each rational $q \in [0, 1]$, $\lim_{n\to\infty} f_n(q)$ exists and there exists a constant $M < \infty$ such that $|f_n'(t)| \leq M$ for all $n \in \mathbf{N}$ and $t \in [0, 1]$. Show that there is a continuous function f such that $(f_n)_{n=1}^{\infty}$ converges uniformly to f.