DOCTORAL EXAM - TOPOLOGY - JUNE 1998

1. Carefully state the Seifert-van Kampen Theorem for the case $X = U \cup V$. Be sure to state all the hypotheses on the spaces involved. Then give a rigorous proof that for any path connected 3-manifold M (without boundary) and any point $x \in M$, one has $\pi_1(M - x) \cong \pi_1(M)$.

2. Give an example of a retract which is not a deformation retract. Prove your answer.

3. Let X be the space obtained by removing the interiors of n disjoint disks from a closed, connected, orientable surface of genus g. Suppose $\pi_1(X)$ is free of rank 4. Compute the Euler characteristic $\chi(X)$. List all the possibilities for the pair (g, n).

4. Find the universal covering spaces of the following spaces. ($X \vee Y$ denotes the space obtained by gluing together X and Y at a single point. Draw a picture in (d).)

   (a) S^1

   (b) $S^1 \times S^1$

   (c) S^2

   (d) $S^1 \vee S^2$

5. Let $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ be chain complexes. Define what it means for $f : \mathcal{C} \to \mathcal{C}'$ to be a chain map. Let $g : \mathcal{C} \to \mathcal{C}'$ be another chain map. Define what it means for f and g to be chain homotopic. Then prove that in this case $f_* = g_* : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$.

6. Suppose X is a space such that $H_0(X) = \mathbf{Z}$, $H_1(X) = \mathbf{Z_6}$, and $H_p(X) = 0$ for $p > 1$. Let P^2 be the real projective plane. Calculate $H_p(X \times P^2)$ for all $p \geq 0$. ($\mathbf{Z_n}$ denotes the cyclic group of order n.)

7. Let M and N be connected n-manifolds (without boundary). Let $M \vee N$ be the union of M and N at a single point. For $p > 0$ express $H_p(M \vee N)$ in terms of $H_p(M)$ and $H_p(N)$, carefully justifying your answer. Do not assume that M and N have triangulations. What is $H_0(M \vee N)$?

8. Suppose M is a compact, orientable, connected 3-manifold (without boundary). $H_p(M) = A_p \oplus T_p$, where A_p is a free abelian group of finite rank and T_p is a finite abelian group. Prove that $A_1 \cong A_2$ and $T_2 = 0$. (Hints: Poincaré Duality and Universal Coefficient Theorem. Recall that every finite abelian group is a direct sum of cyclic groups.)