**Comprehensive Examination**
**Advanced Calculus**
**June 1999**

A passing mark will be given if 5 problems are completely solved.

1. Use the open cover definition of compactness to prove that if a subset $A$ of the real numbers is compact and infinite, then it has an accumulation point.
2. Let $S$ be a non-empty, bounded subset of $\mathbb{R}$. Let $k \in \mathbb{R}$ and define $kS = \{ks : s \in S\}$.
   (a) Prove that if $k > 0$, $\sup kS = k \sup S$.
   (b) State and prove similar results for $\sup kS$ and $k \leq 0$.
3. Suppose that $f : [a, b] \to \mathbb{R}$.
   (a) Define $f$ *is Riemann integrable.*
   (b) Give an example of a bounded function on [0,1] which is not Riemann integrable and prove by using the definition that it is not.
4. Let $I = [0, 1]$. Prove each of the following.
   (a) $\displaystyle\sum_{n=1}^{\infty} x^n(1-x)$ converges pointwise on $I$ but not uniformly.
   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n x^n(1-x)$ converges uniformly on $I$.
5. Suppose that $a < b < c$, $f$ is continuous on $[a, c]$ and $f''(x)$ exists for all $x \in (a, c)$. If $f(a) = a$, $f(b) = b$, and $f(c) = c$, prove that there exists some $d \in (a, c)$ such that $f''(d) = 0$.
6. Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then $f$ is not injective.
7. Suppose that $f(x, y)$ is a real-valued function on $\{(x, y) : a \leq x \leq b, A \leq y \leq B\}$. For each step of the following argument, give a reasonable condition on $f$ which makes the step valid and justify the step. Define

$$g(x) = \int_A^B f(x, y)\, dy.$$

Step 1
$$= \int_A^B \left[ f(a, y) + \int_a^x \frac{\partial f}{\partial x}(t, y) dt \right] dy$$

Step 2
$$= \int_A^B f(a, y)\, dy + \int_A^B \int_a^x \frac{\partial f}{\partial x}(t, y) dt\, dy$$

Step 3
$$= \int_A^B f(a, y)\, dy + \int_a^x \int_A^B \frac{\partial f}{\partial x}(t, y) dy\, dt$$

Step 4
$$\text{Therefore } g'(x) = \int_A^B \frac{\partial f}{\partial x}(x, y) dy$$