Masters Algebra Examination
Oklahoma State University
Department of Mathematics
Summer, 1999

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Let V be a finite-dimensional vector space over a field F. Prove or disprove:
   (a) The intersection of two subspaces of V is a subspace of V.
   (b) The union of two subspaces of V is a subspace of V.

2. Let n be a positive integer and let $p(x) = x(x-1)(x-2)\cdots(x-n)$. Define $n+1$ polynomials of degree n by $p_j(x) = p(x)/(x-j)$ for $0 \le j \le n$.
   (a) Prove that $\{p_0, \ldots, p_n\}$ is a basis of the vector space of real polynomials of degree at most n.
   (b) Find the coordinate vector of $q(x) = x^n$ relative to the basis in part (a).

3. Let $p > 2$ be a prime number. Let G be a group of order $2p$.
   (a) Prove that G has exactly $p-1$ elements of order p.
   (b) Prove that G is either cyclic or isomorphic to the dihedral group of order $2p$.

4. Prove that any finite subgroup of the multiplicative group of a field is cyclic.

5. Determine the simplest expression for the ring

$$R = \mathbb{Z}[\alpha]/(2\alpha - 6, 6\alpha - 15)$$

   (Here (a, b) means the ideal generated by the elements a and b.)

6. Let $K = F(\alpha)$ where F is a field and α is a nonzero root of the irreducible polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in F[x].$$

   Write the element $\alpha^{-1} \in K$ as a polynomial of α with coefficients in F.