Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
Summer, 1999

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. (a) What does it mean for a group to be solvable?
   (b) Let G be a group and H a normal subgroup of G. Show that G is solvable if and only if both H and G/H are solvable.
   (c) For which n is S_n solvable? (You do not need to prove your answer.)

2. Show that the intersection of all prime ideals in a commutative Noetherian ring is nilpotent (i.e. $\mathfrak{a}$ is nilpotent if $\mathfrak{a}^k = 0$ for some integer $k \geq 1$).

3. Let R be a ring.
   (a) State three equivalent criteria for a (left) R-module P to be projective. (You do not need to prove they are equivalent.)
   (b) If $e \in R$ is an idempotent, show that the left ideal Re is projective as an R-module.
   (c) Show that the direct sum of projective modules is projective.

4. Let p be a prime and $q = p^n$ for some $n \geq 1$. Let $\mathbb{F}_q$ be a finite field with q elements.
   (a) What is the prime field of $\mathbb{F}_q$?
   (b) Show that $\mathbb{F}_q$ is Galois over its prime field and determine its Galois group.
   (c) Describe the Galois group explicitly in terms of field automorphisms.

5. Let $f(t) = t^7 - 3 \in \mathbb{Q}[t]$ and let $\alpha \in \mathbb{R}$ be the unique real root of f. Let $K = \mathbb{Q}(\alpha)$.
   (a) Prove that $[K : \mathbb{Q}] = 7$.
   (b) Let $L \subset \mathbb{C}$ be the splitting field of f. Prove that $[L : \mathbb{Q}] = 42$.
   (c) Prove that the Galois group G of L over $\mathbb{Q}$ contains elements σ and τ with the following properties:

$$\sigma^7 = 1, \quad \tau^6 = 1, \quad \tau\sigma\tau^{-1} = \sigma^3, \quad G = \langle \sigma, \tau \rangle.$$

6. Prove the following polynomials are irreducible over $\mathbb{Z}$.
   (a) $x^4 + 1$
   (b) $x^5 - x^2 + 1$