PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—June 1999

Notation: $B = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc

1. Let f and g be analytic functions on B so that $f(z)g(z) = 0$ for all $z \in B$. Show that either $f(z) = 0$ for all $z \in B$ or $g(z) = 0$ for all $z \in B$.

2. Show that there does not exist a sequence of (analytic) polynomials $P_n(z)$ which converges to $1/z$ uniformly on $\{z : |z| = 1\}$.

3. Let γ be the simple closed curve defined by $\gamma(t) = e^{2\pi it}$ for $0 \leq t \leq 1$, and let f be entire.

   (a) Suppose there exists a positive integer n such that

   $$\int_\gamma \frac{f(z)}{(z-\alpha)^n}\, dz = 0$$

   for all $\alpha \in B$. Prove that f is a polynomial.

   (b) Suppose that for each $\alpha \in B$ there exists a positive integer $n(\alpha)$ such that

   $$\int_\gamma \frac{f(z)}{(z-\alpha)^{n(\alpha)}}\, dz = 0.$$

   Prove that f is a polynomial.

4. Suppose that f is analytic on $\mathbf{C}$ and that $|\mathrm{Im} f| \leq |\mathrm{Re} f|$. Show that f is constant.

5. Determine the number of roots of the equation

   $$2z^5 - 6z^2 + z + 1 = 0$$

   in the region $\{z : 1 < |z| < 2\}$.

6. Let $\mathcal{F}$ denote the family of functions f analytic on B so that $f(0) = 0$ and

   $$|f'(z)| \leq \frac{1}{1-|z|}$$

   for all $z \in B$. Show that $\mathcal{F}$ is a normal family.