Comprehensive Exam
Real Analysis
June 1999

A passing mark will be given if 5 problems are completely solved.

1. Let $(X, \mathcal{A})$ be a measurable space.
   (a) Define the notion of a σ-*finite measure* on $(X, \mathcal{A})$.
   (b) Let $X = \mathbb{R}$, $\mathcal{A}$ be the Borel σ-algebra, and μ be Lebesgue measure. Give an example of a measurable set which is not countable, not bounded and has measure 0.

2. Suppose that $f \in L_1([0,1], \mathcal{B}, m)$, where m denotes Lebesgue measure and $\mathcal{B}$ is the Borel σ-algebra. Show that for every $\epsilon > 0$ there is a function g such that $g([0,1])$ is a finite set A, for each $a \in A$, $g^{-1}(a)$ is a finite union of intervals, and $\|f - g\|_1 < \epsilon$.

3. Suppose that $f, g \in L_1(\mathbb{R})$, $g > 0$ a.e., and that $f \geq 0$ a.e. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that if A is measurable and
$$\int_A g < \delta \qquad \text{then} \quad \int_A f < \epsilon.$$

4. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions from $[0,1]$ to $\mathbb{R}$ and there is a finite number M such that $|f_n(x)| \leq M$ and $|f_n'(x)| \leq M$ for all $x \in [0,1]$. Prove that there is a subsequence of $(f_n)_{n=1}^{\infty}$ which converges uniformly on $[0,1]$.

5. Let J be the formal identity map i.e., $Jf = f$. Prove or disprove each of the following.
   (a) J is well defined and continuous from $L_1[0,1]$ to $L_2[0,1]$.
   (b) J is well defined and continuous from $L_2[0,1]$ to $L_1[0,1]$.
   (c) If S is a linear subspace of $L_2[0,1]$ and S is closed in the $L_1[0,1]$ norm, then S is closed in $L_2[0,1]$.

6. For $y \in [0,1]$ and $f \in C[0,1]$ show that the formula
$$Tf(y) = \int_0^1 f(x) \sin(xy)\, dx$$
defines a bounded linear operator T from $C[0,1]$ to itself. (Here $C[0,1]$ has the supremum norm.)