Master's Comprehensive Exam – Advanced Calculus – January 1992

1. (a) Define a sequence p_n of points is a Cauchy sequence.
   (b) Show that any convergent sequence is a Cauchy sequence.
   (c) For this part you are to prove part of the converse of part (b). Assume that p_n is a Cauchy sequence and assume that every bounded monotonic sequence is convergent. Construct the point p which is the limit point of p_n. (Note: Just describe the method by which we know that this point exists – you do not have to show that it is the limit.)
2. (a) Define a set S is n-space is compact.
   (b) Show that any compact set is bounded.
3. (a) Define the double integral $\int\int_R f$ exists and has value v where R is a rectangle.
   (b) Let $f : [a, b] \to \mathbf{R}$ be continuous. Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that if N is a partition of $[a, b]$ with mesh $d(N) < \delta$ then the difference between the upper and lower Riemann sums satisfies $|\overline{S}(N) - \underline{S}(N)| < \epsilon$
4. (a) Define $f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$.
   (b) Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Show that $f(x)x^n$ converges uniformly to 0 for $x \in [0, 1]$.
5. Suppose that $f'(x)$ is continuous on the real line $\mathbf{R}$. Show that for any finite interval $[a, b]$ there is a constant K such that if $x \in [a, b]$ and $y \in [a, b]$ then $|f(x) - f(y)| < K|x - y|$.
6. Does the following series converge pointwise on the reals $\mathbf{R}$? Does it converge uniformly on $\mathbf{R}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$$

7. Consider $F(x, y, z) = -x^2 + 3y^2 + 2y^3z^2 = 0$. If $p_0 = (1, -1, 1)$ then $F(p_0) = 0$.
   (a) Does $F(x, y, z) = 0$ define a function $z = \phi(x, y)$ in some neighborhood of $(x, y) = (1, -1)$ which satisfies $F(x, y, \phi(x, y)) = 0$? Why?
   (b) Does $F(x, y, z) = 0$ define a function $y = \psi(x, z)$ in some neighborhood of $(x, z) = (1, 1)$ which satisfies $F(x, \psi(x, z), z) = 0$? Why?