Masters Comprehensive Exam
Advanced Calculus
January 1993

Each question is worth 10 points.

1. Give an example of a subset $D \subset \mathbf{R}$ and a continuous function f with domain D such that f is not uniformly continuous on D. Prove your assertion that f is continuous but not uniformly continuous.

2. Let n be a non-negative integer, and $P(x)$ a polynomial in x. Consider the function

$$f(x) = \begin{cases} \frac{P(x)e^{(-1/x^2)}}{x^n} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $f \in C^1$ and $f'(0) = 0$.

   (b) Show that $f \in C^\infty$.

   (c) Show that f is not analytic unless $P(x) = 0$.

3. Consider the function $u = f(x, y)$, where $x = s^2t$ and $y = st$. Here, s, t are independent variables, and $f \in C^2$. Find u_{st} in terms of the partial derivatives of f.

4. True or false:

   (a) If $f_n(x)$ is a continuous function of $x \in \mathbf{R}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise, then $f(x)$ is continuous.

   (b) With the same hypotheses as in part (a), $\lim_{n\to\infty} \int f_n = \int f$.

   For each part: if the answer is "true", give a proof. If the answer is "false", give a counterexample, with justification.

5. Let $C \subset \mathbf{R}^n$.

   (a) Define what it means for C to be *compact.*

   (b) Prove that if C is compact, then it is closed and bounded.

6. Let f be a function defined on a set $D \subset \mathbf{R}^n$.

   (a) Define $\int_D f$.

   (b) Give as many conditions as you can on D and f for this integral to exist.

7. Prove that if a series converges absolutely, then it converges.

8. Let a_n be a sequence of real numbers. Give the definitions of $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$. Prove that if they are equal and finite, then $\lim_{n\to\infty} a_n$ exists, and is equal to this common value.

9. Let C be the curve bounding the semicircular region R in $\mathbf{R}^2$ defined by the inequalities $y \geq 0,\ x^2 + y^2 \leq 1$. Evaluate the integral $\int_C y(dx + dy)$ in two ways:

   (a) Directly from the definition of the line integral.

   (b) As an integral over R, using Green's theorem.

   The curve C is to be traversed in the counterclockwise direction.