Masters Comprehensive Exam

Modern Algebra

January 1993

1. (a) If G is a group of order p^n, with p a prime, show that G has a non-trivial center.

(b) If G is a group of order p^2, with p a prime, show that G is abelian.

(c) Are all groups G of order p^3, with p a prime, abelian? (Prove or give a counter-example.)

2. (a) List all abelian groups of order 28 (up to isomorphism).

(b) Can a group of order 28 be simple? Prove your answer.

(c) Construct 2 non-abelian groups of order 28. How many non-abelian groups of order 28 do you think there are?

3. Let R be a commutative ring with unit. Recall that an ideal $P \subset R$ is called prime if $ab \in P$ implies $a \in P$ or $b \in P$.

(a) Show that any maximal ideal of R must be prime.

(b) If R is a PID, show that all prime ideals are maximal.

(c) Give an example of a commutative ring R and an ideal $I \subset R$ such that I is prime but not maximal.

4. Let M be a free module of finite rank over a PID R.

(a) Show that every submodule $N \subset M$ is also free.

(b) Is it true that every quotient module of the free module M is also free? If so, prove. If not, give a counter example.

5. (a) If V is a finite dimensional vector space over a field F and V^* is its dual vector space, show that $V \simeq V^*$.

(b) Give an example of a vector space over $\mathbb{Q}$ which is not finite dimensional.

6. (a) Compute the Galois group G of $f(x) = x^3 - 5$ over $\mathbb{Q}$.

(b) If $F \supset \mathbb{Q}$ is the splitting field of this polynomial, write down all intermediate fields $F \supset E \supset \mathbb{Q}$.

(over)

7. Let $F \supset E$ be an extension of fields.

(a) Define what it means for F to be an algebraic extension of E.

(b) If $(F : E) < \infty$, show that F is an algebraic extension of E.

(c) Do infinite algebraic extensions exist? If not, prove it. If so, give an example.