PH. D. COMPREHENSIVE EXAM
ALGEBRA
JANUARY, 1993

1. Pair each group in column A with an isomorphic group in column B. Here, $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$, $\mathbf{Z}_n^\times$ = group of units with respect to multiplication in $\mathbf{Z}_n$, S_n = symmetric group on n objects, A_n = alternating group on n objects.

A	B
$\mathbf{Z}_6 \times \mathbf{Z}_2$	$\mathbf{Z}_{15} \times \mathbf{Z}_{12}^\times$
$\mathbf{Z}_{15}^\times$	$\mathbf{Z}_5^\times \times (263\mathbf{Z}/526\mathbf{Z})$
$\mathbf{Z}_{30} \times \mathbf{Z}_2$	$\mathbf{Z}_2 \times A_3 \times (S_{20}/A_{20})$
$\mathbf{Z}_8^\times \times \mathbf{Z}_{15}$	$\mathbf{Z}_{11}^\times \times \mathbf{Z}_7^\times$
$\mathbf{Z}_8$	$\mathbf{Z}_{64}/8\mathbf{Z}_{64}$

2. Prove or disprove: Every group of order 27 is abelian.

3. Prove or disprove: $\mathbf{Z}[x]$ is a principal ideal domain.

4. Let R be a ring with a two-sided ideal I. Suppose there exists $e \in I$ such that $ei = ie = i$ for all $i \in I$.
(a) Show that e lies in the center of R.
(b) Show that there exists a two-sided ideal J of R such that $R = I \oplus J$.

5. Factor the polynomial $x^5 - 1$ completely into irreducible factors over the following fields and justify your answer: (a) $\mathbf{Q}$ (b) $\mathbf{R}$ (c) $\mathbf{C}$ (d) $\mathbf{Z}/11\mathbf{Z}$ (e) $\mathbf{Z}/13\mathbf{Z}$.

6. In each of parts (a) through (e) of problem 5, compute the Galois group of the splitting field of $x^5 - 1$ over the indicated field and justify your answer.