Ph.D. COMPREHENSIVE EXAM
REAL ANALYSIS
January 1994

Justify your answers. Four complete solutions will constitute a passing score.

1. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X.
   a. Suppose that f is a function from X into the extended real numbers. Define the phrase f *is measurable.*
   b. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at each point of $\mathbb{R}$. Prove that f' is Borel measurable.

2. With the same notation as in Problem 1, define a set $A \in \mathcal{A}$ to be an *atom* of a measure μ on $(X, \mathcal{A})$ if $\mu(A) > 0$, and for all $B \subset A$ such that $B \in \mathcal{A}$, we have $\mu(B) = \mu(A)$ or $\mu(B) = 0$.
   a. Suppose that f is a right-continuous non-decreasing real-valued function on $[0, 1]$. Prove that x_0 is a discontinuity of f if and only if $\{x_0\}$ is an atom of the induced measure μ_f. ($\mu_f(\,(a, b]\,) = f(b) - f(a)$, for all $a, b \in [0, 1]$.)
   b. Give an example of a strictly increasing function g on [0,1] such that $g' = 0$ almost everywhere with respect to Lebesgue measure.
   c. Give an example (sketch the construction and verification) of a non-decreasing function h on [0,1] such that $h' = 0$ almost everywhere with respect to Lebesgue measure but the induced measure μ_h has no atoms.

3. a. Define the term *Hilbert space.* (Assume the scalar field is $\mathbb{C}$.)
   b. Define *orthonormal system* and *complete orthonormal system* in a Hilbert space H.
   c. Let $\rho_n(t) = \operatorname{signum}(\sin(2^n \pi t))$ for $t \in [0, 1]$ and $n = 0, 1, 2, \ldots$. Show that $(\rho_n)_n$ is an orthonormal system in $L^2[0, 1]$ (with Lebesgue measure). ($\operatorname{signum}(x) = x/|x|$ if $x \neq 0$ and $\operatorname{signum}(x) = 0$ if $x = 0$.)
   d. Prove that for all $f \in L^2[0, 1]$, $\lim_{n\to\infty} \int f \rho_n = 0$.

4. a. Suppose that K is a compact metric space, state the Riesz Representation Theorem for $C(K)^*$.
   b. Prove that for all $f \in C[0, 1]$,
   $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} f\left(\frac{k}{n}\right) \text{ exists.}$$
   c. Prove that there is a Borel measure μ on [0,1] such that
   $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} f\left(\frac{k}{n}\right) = \int f \, d\mu$$
   for all $f \in C[0, 1]$.

5. a. Suppose that f is a continuous real-valued function on (0,1], $f \geq 0$ and the improper Riemann integral of f on (0,1] is finite. Prove that f is Lebesgue integrable on [0,1].
   b. Prove or disprove: The assumption $f \geq 0$ is necessary.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX