Master's Comprehensive Exam – Advanced Calculus – January 1995

Part I: Do <u>all</u> of these problems.

1. (5 points) Define $f : [a, b] \to R$ is a Riemann integrable function.

2. (5 points) Define the function $s : [0, 3] \to R$ to be the greatest integer function:
$$s(x) = \left\{ \begin{array}{ll} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & x = 3 \end{array} \right\}$$
Define $A(x) = \int_0^x s(t)\,dt$. At what points in [0,3] is $A(x)$ continuous? At what points is it differentiable?

3. (16 points)
   (a) Define a Cauchy sequence.
   (b) Define uniform continuity of a function.
   (c) Show that if the sequence $\{p_n\}$ is Cauchy and f is a continuous function, then the sequence $\{f(p_n)\}$ is also Cauchy,
   (d) Give an example of a continuous function and a Cauchy sequence $\{p_n\}$ such that $\{f(p_n)\}$ is not Cauchy.

4. (a) (6 points) For what values of x does the series $\sum_{n=1}^{\infty}(1 - x^2)^n$ converge?
   (b) (6 points) Does this series converge uniformly on $(\frac{1}{2}, 1)$? Why?

5. (10 points) Suppose that $f : [0, 5] \to R$ is continuous and $f(0) = 1$. Also suppose f is differentiable and $|f'(x)| \le 3$ for $x \in (0, 5)$. Show that $f(5) < 17$.

6. (8 points) Let $B = B(\mathbf{0}, r)$ denote the open ball of radius $r > 0$ centered at the origin $\mathbf{0}$ in R^n. Suppose $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ defines a function $f : B \to R^n$ and that each of the component functions f_i has continuous partial derivatives in B. Suppose that for each vector $x \in B$, $f(x) = f(-x)$. Why is it true that the Jacobian determinant $J_f(\mathbf{0})$ must be zero at $x = \mathbf{0}$?

7. (10 points) Consider $f(x, y, z)$ defining a function $f : R^3 \to R$. Given that the directional derivative of f in the direction $\vec{u}$ is given by the dot product $\nabla f \cdot \vec{u}$, discuss the following.
   (a) Why it is that the gradient vector ∇f always points in the direction of maximal directional derivative?
   (b) Why is it that level curves of the surface $f(x, y, z)$ always run orthogonal to the gradient?

8. (10 points) Suppose that $F(x, y) = f(xe^y, xy)$ where f is a real-valued function of two variables. Express $F_1 = \frac{\partial F}{\partial x}$, $F_2 = \frac{\partial F}{\partial y}$ and $F_{1,2} = \dfrac{\partial F^2}{\partial x \partial y}$ in terms of the partials of f.

Part II: Do <u>only two</u> of the last three problems.

9. (12 points) Suppose that $f_n : [0, 1] \to R$ is a sequence of continuous functions and $\{f_n\}$ converges uniformly to f. Show that f is continuous on $[0, 1]$.

10. (12 points) Show that the set of all sequences of 0's and 1's is an uncountable set.

11. (12 points)
    (a) Define a set S is n-space is compact.
    (b) Show from the definition that any compact set is closed. (Hint: A possible approach is to use the fact that if a set is not closed, then there is a point p not in the set which is an accumulation point of the set.)