Masters Comprehensive Exam - Modern Algebra - January 1995

1. List all Abelian groups of order 36 up to isomorphism (list each group exactly once).

2. Consider groups G of order 18.

   (a) Give all primes p for which G has a p-Sylow subgroup.

   (b) For each such p, give the order of the p-Sylow subgroups, and the possibilities for the number of p-Sylow subgroups.

   (c) For each possible number of p-Sylow subgroups, give an example of a group G of order 18 with that number of p-Sylow subgroups, and list these subgroups in each example.

   (d) Which of these subgroups are normal?

3. (a) Prove that an $n \times n$ real matrix A is diagonalizable if and only if $\mathbf{R}^n$ has a basis consisting of eigenvectors for A.

   (b) Give an example of a 3×3 matrix that is *not* diagonalizable, and fully explain why not.

4. Let R be the ring consisting of all expressions $a + b\alpha + c\beta + d\gamma$, where a, b, c, d are complex numbers. Here addition is defined by

$$(a + b\alpha + c\beta + d\gamma) + (a' + b'\alpha + c'\beta + d'\gamma) =$$

$$(a + a') + (b + b')\alpha + (c + c')\beta + (d + d')\gamma,$$

   and multiplication is determined by the relations $\alpha^2 = \beta^2 = \gamma^2 = -1,\ \alpha\beta = -\beta\alpha = \gamma,\ \beta\gamma = -\gamma\beta = \alpha,\ \gamma\alpha = -\alpha\gamma = \beta$, $\mathbf{C}$-linearity and the distributive law. In other words, R is essentially the same as the quaternions, except that coefficients are complex rather than real, and α, β, γ have been used in place of i, j, k to avoid confusion with any i's that occur in the complex coefficients a, b, c, d.

   (a) Show that R is isomorphic as a ring to the ring of 2×2 complex matrices.

   (b) Are the quaternions a division ring? Explain.

   (c) Is R a division ring? Explain.

5. In this question, $\omega \in \mathbf{C}$ denotes a primitive third root of unity.

   (a) Is $\mathbf{Q}(\sqrt[3]{2})$ a Galois extension of $\mathbf{Q}$? Explain.

   (b) Find the splitting field K of the polynomial $x^3 - 2$ over $\mathbf{Q}$.

   (c) Identify the Galois groups of $K/\mathbf{Q}$ and $K/\mathbf{Q}(\omega)$ with more familiar groups.

   (d) Find all intermediate fields between $\mathbf{Q}$ and K.