Ph.D. COMPREHENSIVE EXAM - TOPOLOGY - JANUARY 1995

1. (12 points) Prove the "tube lemma": Let X and Y be topological spaces. Give $X \times Y$ the product topology. Assume Y is compact. Let $x_0 \in X$. Let N be an open subset of $X \times Y$ which contains the "slice" $\{x_0\} \times Y$. Then there is an open neighborhood W of x_0 in X such that the "tube" $W \times Y$ is contained in N.

2. (12 points) Let X be a topological space which satisfies the second axiom of countability. Let A be an uncountable subset of X. Prove that A has a limit point in X.

3. (12 points) Let (X, d) be a metric space. Let $x_0 \in X$. Define $f : X \to \mathbf{R}$ by $f(x) = d(x, x_0)$. Prove that f is continuous.

4. (12 points)

   (a) State the Urysohn Lemma.

   (b) Prove that every connected normal space having at least two points is uncountable.

5. (12 points) Let $x_0 \in A \subseteq X$. Suppose $r : X \to A$ is a retraction.

   (a) Prove that $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is onto.

   (b) Prove that if r is a deformation retraction, then r_* is one-to-one.

   (c) Give an example of a retraction that is not a deformation retraction.

6. (12 points) Let $G = G_1 * G_2$.

   (a) Prove that if G_1 and G_2 are free groups, then G is a free group.

   (b) Is the converse true? (No proof required.)

7. (12 points) Let $p : \widetilde{X} \to X$ be a covering space. $\widetilde{X}$ and X are assumed to be path connected and locally path connected. Let $\tilde{x}_0 \in \widetilde{X}$. Let $x_0 = p(\tilde{x}_0)$.

   (a) Prove that $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is one-to-one.

   (b) Let $\tilde{x}_1 \in p^{-1}(x_0)$. Prove that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}_1))$ are conjugate subgroups of $\pi_1(X, x_0)$.

8. (12 points) Give presentations for the fundamental groups of the following spaces. P denotes a projective plane.

   (a) $P \times P$

   (b) $P \# P$, where "$\#$" denotes connected sum.

   (c) $P \vee P$, where "$\vee$" denotes the gluing of the two spaces together at a single point.

   (d) $P - \{x\}$, where $x \in P$.

9. (4 points) State the classification theorem for compact surfaces.