Master's Comprehensive Exam
Advanced Linear Algebra
January 1996

1. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -2 & 2 & 1 \\ 2 & 4 & 1 & 9 \end{bmatrix}$ Find the bases for the following:

   (a) The row space of A.

   (b) The column space of A.

   (c) The nullspace of A.

   (d) The left nullspace of A (i.e. the set of all x such that $xA = 0$.)

2. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

   (a) Find the characteristic polynomial $c(x)$ and the polynomial $m(x)$ of A.

   (b) Find the eigenvalues of A and a basis for each eigenspace.

   (c) Determine whether or not A is diagonalizable; if it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

3. Let V be a finite dimensional real vector space with an inner product $\langle , \rangle$. Let W be a proper subspace of V. Let $W^{\perp} = \{v \in V | \langle v, w \rangle = 0 \forall w \in W\}$. Prove that $V = W \oplus W^{\perp}$.

4. Let V be a vector space over a field F. Let V^* be its dual space.

   (a) Suppose $T : V \to V$ is a linear transformation. Define the transpose $T^* : V^* \to V^*$.

   (b) Now suppose V has a finite basis $\mathcal{B} = \{v_1, ..., v_n\}$. Define the dual basis $\mathcal{B}^*$ for V^*.

   (c) Suppose A is the matrix of T with respect to $\mathcal{B}$. What is the matrix of T^* with respect to $\mathcal{B}^*$? Prove your answer.

5. Suppose A and B are complex $n \times n$ matrices having the same characteristic polynomial and the same minimal polynomial.

   (a) Prove that if $n = 3$, then A and B are similar over $\mathbf{C}$.

   (b) Is this true for $n = 4$? Give a proof or a counterexample.

   (c) Answer (a) and (b) for real matrices and similarity over $\mathbf{R}$.