MASTER'S COMPREHENSIVE EXAM—TOPOLOGY—JANUARY 1996

1. Prove that every metric space is normal.

2. X is a topological space. There exists a basis $\mathcal{B}$ for the topology on X such that any two distinct members of $\mathcal{B}$ are disjoint. What's more, X is Hausdorff. Prove that X has the discrete topology.

3. (a) Let X be a Hausdorff space. Let K_1, K_2 be compact subsets of X. Prove that $K_1 \cap K_2$ is compact.

(b) Give a counterexample when X is not Hausdorff. Hint: let T be a space with two points in the indiscrete topology. Consider $\mathbb{R} \times T$.

4. Let $\{X_\alpha\}$ be a family of topological spaces, with α ranging through an index set J. For all α, let Y_α be an open subset of X_α. Consider the space X whose points are J-tuples $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ such that

(1) $x_\alpha \in X_\alpha$ for every $\alpha \in J$, and
(2) $x_\alpha \in Y_\alpha$ for all but finitely many $\alpha \in J$. (Which α are left out depends on $\mathbf{x}$.)

We give X a topology by taking as a basis the sets

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α for all α, and $U_\alpha = Y_\alpha$ for almost all α. With this topology, we call X the *restricted topological product* of the X_α with respect to the Y_α.

(a) Fix a finite set $S \subset J$. Let X_S be the set of all $\mathbf{x} \in X$ such that $x_\alpha \in Y_\alpha$ whenever $\alpha \notin S$. Explain why X_S is open in X.

(b) Notice that, as a set,

$$X_S = \prod_{\alpha \in S} X_\alpha \times \prod_{\alpha \notin S} Y_\alpha. \tag{4.1}$$

So it is natural to give X_S the product topology coming from (4.1). On the other hand, since $X_S \subseteq X$, it has a subspace topology coming from X. Prove that this subspace topology and that product topology are the same.

(c) Assume in addition that each X_α is locally compact and that each Y_α is compact. Prove that X is locally compact. Hint: use the X_S.