Master's Comprehensive Exam – Advanced Calculus – January 1997

1. Define the double Riemann integral $\int\int_A f$ for a continuous function f defined on a rectangle A in R^2.

2. Suppose that $u = f(x, y)$ where f and all of its partial derivatives exist. If $x = s^2 + t$ and if $y = st$ then find u_{st} in terms of the partial derivatives of f.

3. (a) Define:
   i. A set S in R^n is open.
   ii. A set S in R^n is closed.
   iii. A point $x \in R^n$ is an accumulation point of a set S.

   (b) Show that a set is closed if and only if it contains all of its accumulation points.

4. Indicate which of these functions are uniformly continuous on the set $(0, \infty)$. No reasoning is needed, just the answer.

(a) $f(x) = x$	(b) $f(x) = \lvert x \rvert$	(c) $f(x) = \frac{1}{x}$
(d) $f(x) = x^2$	(e) $f(x) = e^x$	(f) $f(x) = \sin \frac{1}{x}$
(g) $f(x) = \frac{1}{x^2+1}$	(h) $f(x) = \sqrt{x}$	(i) $f(x) = \sin x$

5. (a) Let $f_n : R \to R$ be a sequence of functions and let A be a subset of R. Let the function f be defined on the set A. Define the concept of the sequence $\{f_n\}$ converging uniformly on the set A to f.

   (b) Suppose that, for each n, $f_n : R \to R$ is continuous and suppose that f_n converges uniformly on R to f. Show that, for any $a \in R$, f is continuous at a.

6. (a) For A a subset of a metric space (S, d_S), give the definition of the set A being compact.

   (b) Let $f : S \to T$ be a continuous function from one metric space (S, d_S) into another (T, d_T). If A is a compact subset of S show that $f(A)$ is a compact subset of T.

7. Suppose that $f : [0, \infty) \to R$ is decreasing and $f(x) > 0$ for all x. Assume there is a constant K such that for each $b > 0$,

$$\int_0^b f(x)\, dx \leq K.$$

   Prove that $\sum_{n=0}^{\infty} f(n)$ converges.

8. Suppose that $f : R \to R$ is continuously differentiable and for some positive number p, $f'(x) \geq p$ for all $x \in R$. Suppose also that $f(0) = 0$ (this is not necessary, but may make calculations easier). Show that

   (a) f is one-to one.

   (b) f is onto.

9. Which of these series converge uniformly on the entire line R? Which converge uniformly on $[0, 1]$? Why?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2 + n^2}{n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{n^2}$