Master's Comprehensive Exam
Advanced Linear Algebra
January 1997

Do only four of the five problems.

1. Consider the vector space V of all polynomials of degree less than or equal to 5. For the linear map $T = \frac{d^2}{dx^2} + 1$ find the Jordan form **AND** give a basis of V with respect to which the matrix is the Jordan form.

2. Let
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$
Find a polynomial $P(x)$ so that $A^{-1} = P(A)$. (Hint: Think about characteristic and/or the minimal polynomial.)

3. Let V be a finite dimensional vector space over an arbirtary field. Let $T : V \to V$ be an arbitrary linear transformation.

   (a) Give the definition of the matrix of T with respect to a basis.

   (b) Prove that the matrices of T with respect to two different bases are similar (i.e. conjugate).

4. Suppose a 7×7 matrix A has minimal polynomial $m(x) = (x^3 - 1)^2$.

   (a) Give the possible rational canonical forms which A can have when the base field is Q.

   (b) Give the possible rational canonical forms which A can have when the base field is C.

5. Let V be an inner product space. For any vector $w \neq 0$ recall that the *reflection* in the hyperplane orthogonal to w is the linear transformation $T_w(x) = -\frac{2(x,w)}{(w,w)}w$.

   (a) Show that the inverse of a reflection is a reflection.

   (b) Show that a reflection is an orthogonal linear transformation.

   (c) Determine the minimal polynomial, the determinant and the Jordan form of an arbirtary reflection.

   (d) Is the composition of two reflections a reflection?