PH. D. COMPREHENSIVE EXAM — ALGEBRA — JANUARY, 1997

Notation: The symbol $\mathbf{Q}$ denotes the field of rational numbers. All rings are assumed to have an identity element.

1. Let G be a group of order $175 = 5^2 \cdot 7$. Show that the 7-Sylow subgroup is contained in the center of G.

2. Let K be a field and U a finite subgroup of the multiplicative group $K^\times \stackrel{\text{def}}{=} K - \{0\}$. Prove that U is cyclic.

3. Let R be a commutative ring and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Prove that M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

4. Let R be a ring and let $I \subseteq R$ be a two-sided ideal. Suppose there exists $e \in I$ such that $ea = ae = a$ for all $a \in I$.
(a) Prove that $xe = ex$ for all $x \in R$.
(b) Show there exists a two-sided ideal J of R such that $R = I \oplus J$.

5. Let F be the splitting field of $x^4 - 7$ over $\mathbf{Q}$.
(a) Find the Galois group of F over $\mathbf{Q}$.
(b) Find all fields K, $\mathbf{Q} \subseteq K \subseteq F$, that are normal over $\mathbf{Q}$.

6. (a) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$.
(b) Find the Galois group of the splitting field of the polynomial from part(a) over $\mathbf{Q}$.