PhD/EdD Comprehensive Exam
Complex Analysis
January 1997

Do any six of the following problems. Be sure to give complete proofs and explanations. A *domain* is always a non-empty open subset of $\mathbb{C}$ (or the Riemann sphere S^2) while a *region* is a non-empty, open and connected.

1. (a) Show that the real part of an analytic function $f(z)$ on a domain Ω is harmonic.

   (b) Is $u(x,y) = \log(x^2+y^2)$ harmonic in $\Omega = \mathbb{C}^\times$? Is it the real part of an analytic function on $\mathbb{C}^\times$? (Here $\mathbb{C}^\times$ denotes the complex plane with the origin removed.)

2. Let $\mathfrak{H} = \{z = x+iy | y > 0\}$ be the upper half plane in $\mathbb{C}$. Show that every one-to-one and onto analytic map $f : \mathfrak{H} \to \mathfrak{H}$ is a linear fractional transformation with real coefficients.

3. Let S^2 denote the Riemann sphere (or extended complex plane). Prove that the only functions that are meromorphic on all of S^2 are the rational functions.

4. Let Ω be a region in $\mathbb{C}$ and f an analytic function on Ω. Show that the following two statements are equivalent.

   (a) $f(z)$ is a polynomial.

   (b) There is a point $a \in \Omega$ such that all derivatives $f^{(n)}(a) = 0$ for all but finitely many n.

5. (Euler, 1743) Evaluate $\displaystyle\int_0^\infty \frac{dz}{1+x^n}$ for $n \geq 2$.

6. Show that the family of functions $\mathcal{F} = \{f \in \mathcal{H}(\Omega) | \operatorname{Re}(f) > 0\}$ on a region Ω is a normal family. (Here $\mathcal{H}(\Omega)$ is the set of all analytic functions on Ω.)

7. Let $\{a_n\}$ be a discrete set of points in $\mathbb{C}$ satisfying

$$\sum_n \frac{1}{|a_n|^{h+1}} < \infty$$

   for some integer $h \geq 0$. Show that there exists an entire function of finite order having zeros precicely at the a_n. What is the general form of such functions?

8. Let S be the open sector defined by $S = \{z \in \mathbb{C} | 0 < |z| < 1 \text{ and } 0 < \arg(z) < \frac{\pi}{2}\}$. Is this set conformally equivalent to the unit disc $\Delta = \{z||z| < 1\}$? If so, explicitly determine a conformal equivalence and determine the image of the intersection of S with the ray $R = \{z|\arg(z) = \frac{\pi}{4}\}$. If not, prove why not.

9. (a) Show that any continuous real valued function $u(x,y)$ which satisfies the Mean Value Property in a region Ω is harmonic in Ω.

   (b) Use (a) to prove the Schwartz Reflection Principle for harmonic functions is the following simple form: If $u(x,y)$ is a continuous real valued function on the half disc $\{z||z| < 1 \text{ and } \operatorname{Im}(z) \geq 0\}$ such that u is harmonic on $\{z||z| < 1 \text{ and } \operatorname{Im}(z) > 0\}$ and $u(x,y) \equiv 0$ on $\{z||z| < 1 \text{ and } \operatorname{Im}(z) = 0\}$ then $u(x,y)$ has a harmonic extension to all of $\Delta = \{z||z| < 1\}$.