Ph.D. COMPREHENSIVE EXAM
REAL ANALYSIS
January 1997

Justify your answers. Four complete solutions will constitute a passing score.

1. Let $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras of subsets of topological spaces X and Y, respectively.
   (a) Suppose that f is a function from X into Y. Define the phrases f *is measurable* and f *is Borel measurable.*
   (b) Suppose that f is a function from X into the real numbers. Define the phrase f *is measurable.*
   (c) Using only the definitions prove that if f and g are measurable real-valued functions on X then $f + g$ is measurable.

2. Let λ denote Lebesgue measure on $\mathbb{R}$. Prove or disprove:
   (a) $L^2(\lambda) \subset L^\infty(\lambda)$.
   (b) $C_0(\mathbb{R})$ is dense (in the norm topology) in $L^\infty(\lambda)$.

3. (a) Define absolute continuity for measures and state the Radon-Nikodym Theorem.
   (b) For each measurable subset A of $\mathbb{R}$ let
   $$\mu(A) = \int_A \exp(-x^2)\, d\lambda(x),$$
   where λ is Lebesgue measure. Show that $\lambda << \mu$ and determine $\frac{d\lambda}{d\mu}$.

4. (a) Define the *weak topology* of a Banach space X.
   (b) Suppose that x and $(x_n)_{n=1}^\infty$ are elements of ℓ^2, where $\ell^2 = L^2(\mu)$ and μ is counting measure on the natural numbers $\mathbb{N}$. Prove that if $(x_n)_{n=1}^\infty$ converges to x weakly then $\lim_n x_n(k) = x(k)$ for all $k \in \mathbb{N}$.
   (c) Prove that if in addition to the assumptions in the previous part $\lim_n \|x_n\| = \|x\|$, then $\lim_n \|x_n - x\| = 0$.

5. Suppose that f is integrable on $[0, \infty)$ (with respect to Lebesgue measure). Define for $x \in \mathbb{R}$,
   $$g(x) = \int_0^\infty f(t) \exp(-|x|t)\, dt.$$
   (a) Prove that g is measurable.
   (b) If f is zero on $[0, \epsilon]$ for some $\epsilon > 0$, show that g is Lebesgue integrable.