Comprehensive Examination
Advanced Calculus
January 1998

A passing mark will be given if 5 problems are completely solved.

1. Let S be a metric space.
   (a) Define *interior* of a set $T \subset S$, $\mathrm{int}(T)$.
   (b) Prove that if $(S_\alpha)_{\alpha \in \mathcal{A}}$ is a sequence of subsets S, then
$$\cup_{\alpha \in \mathcal{A}} \ \mathrm{int}(S_\alpha) \subset \ \mathrm{int}(\cup_{\alpha \in \mathcal{A}} S_\alpha).$$
2. Show that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable, with $|f'(x)| \leq 1$ for all $x \in \mathbf{R}$ and $f(0) = 0$, then $|f(x)| \leq |x|$ for all $x \in \mathbf{R}$.
3. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers. Prove that $\liminf_{n\to\infty} na_n = 0$.
4. Determine the radius of convergence and the convergence at the endpoints of the interval of convergence for the power series $\sum_{n=1}^{\infty} a_n x^n$ where
$$a_n = \begin{cases} \dfrac{1}{n2^n} & \text{if } n \text{ is even,} \\ \dfrac{3^n}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$
5. Let $f_n(x) = nxe^{-nx}$ for $n = 1, 2, 3, \ldots$.
   (a) Show that for any $\delta > 0$, $(f_n)_{n=1}^{\infty}$ converges uniformly to the function 0 on $[\delta, \infty)$.
   (b) Show that $(f_n)_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$.
6. Let
$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else.} \end{cases}$$
   (a) For each unit vector $\mathbf{u}$ calculate the directional derivative of f at $(0, 0)$ in the direction $\mathbf{u}$.
   (b) Explain why f is not differentiable at $(0, 0)$.
7. Evaluate
$$\int_C (y^2 + \log(1 + x^2))\, dx + (e^{\sin y} - x^2)\, dy$$
where C is the boundary of the unit square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ oriented counterclockwise.