Masters Algebra Examination
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1. Let G be a finite group and x and y elements of G. Let a be the order of x and b be the order of y. It is given that a and b are relatively prime.
   (a) Prove: if G is abelian, then the subgroup generated by x and y has order ab.
   (b) Prove: if G is abelian, then the subgroup generated by xy has order ab.
   (c) By way of example, show that (a) and (b) do not necessarily hold in a non-abelian group.

2. Consider S_n, the symmetric group on n objects.
   (a) Show that any two r–cycles in S_n are conjugate.
   (b) If $(a_1\ a_2\ \cdots\ a_r)$ is an r–cycle in S_n, what is the size of its conjugacy class?
   (c) Determine the center of S_n.

3. Let V be a vector space over a field F, and let $T : V \to V$ be a linear transformation of V into itself. Suppose there are non-zero vectors $v_1, \ldots, v_m$ and *distinct* scalars $c_1, \ldots, c_m$ (i.e., $c_i \neq c_j$ for $i \neq j$) such that $T(v_j) = c_j v_j$ for each $j = 1, \ldots, m$. Prove that the vectors $v_1, \ldots, v_m$ are linearly independent.

4. Let R be an integral domain together with a function $\sigma : R \to \{0, 1, 2, 3, ...\}$ satisfying:
   (a) $\sigma(a) = 0$ if and only if $a = 0$;
   (b) For any a,$b \in R$, $a \neq 0$, either $b = qa$ for some $q \in R$ or there are x,$y \in R$ such that $0 < \sigma(ax + by) < \sigma(a)$.

   Prove R is a principal ideal domain.

5. (a) Let $K \subset F \subset E$ be a tower of field extensions of finite degree. Prove that $[E : K] = [E : F]\,[F : K]$.
   (b) Let E/F be a field extension of finite degree, and suppose $E = F(\alpha)$ for some $\alpha \in E$. If the degree $[E : F]$ is odd, prove that $E = F(\alpha^2)$.

6. Let Λ be the $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ generated by $(2, 3, -1)$ and $(1, 0, -1)$. Express $\mathbb{Z}^3/\Lambda$ as a direct product of cyclic groups.