Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
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1. Let G be a group, let $[G, G]$ be the commutator subgroup of G, and let N be the subgroup of G generated by the set

$$K = \{x^2 : x \in G\}.$$

   (a) Prove that N is a normal subgroup of G, and that $[G, G] \subset N$.

   (b) Give an example of a group G with $[G, G] \neq N$.

   (c) Give an example of a group G with $K \neq N$.

2. Show that $\mathbb{Z}[x]$ is a unique factorization domain. Is it a principal ideal domain? Prove your answer.

3. Let R be a ring.

   (a) Define what it means for a R-module P to be **projective** in terms of exact sequences of R-modules

$$M' \to M \to 0$$

   and module homomorphisms $f : P \to M$.

   (b) Using the definition from (a), prove that an R-module P is projective if and only if P is a direct summand of a free R-module.

4. Let K/F be an extension of fields and f an irreducible polynomial over F. Prove that if u and v are roots of f in K then there is an F-homomorphism $\sigma : F(u) \to F(v)$. Show by example that the result may be false if f is not irreducible.

5. Let $f(x) = x^4 - 4$.

   (a) Determine the degree of the splitting field K of $f(x)$ over $\mathbb{Q}$.

   (b) Determine all the subextensions F such that $\mathbb{Q} \subset F \subset K$, and arrange these in a graph ordered by inclusion.

6. Determine all possible operations of the symmetric group S_3 on a set of 4 elements. (Two such operations will be considered the same if by relabelling the 4 elements the operations become identical.)