PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 1998

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. (a) State Cauchy's Theorem (in the most general form you know).

   (b) Give an example of an open set $\Omega \subset \mathbf{C}$, a function f analytic on Ω, and a smooth closed curve γ in Ω such that $\int_\gamma f(z)\, dz \neq 0$. Explain why this example does not contradict Cauchy's Theorem.

2. Let f be an entire function, and suppose there is a finite constant M so that

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq M$$

   for all $r > 0$. Show that f is constant by finding $f^{(n)}(0)$ for all $n \geq 1$.

3. Let D be a connected open set in $\mathbf{C}$ and f a function analytic on D.

   a) Assuming that f has an analytic logarithm in D, show that f has analytic mth roots for all m. Explicitly: Assume that there exists h analytic on D so that $f = e^h$, and show that for each $m \in \mathbf{N}$ there exists a function g_m analytic on D so that $f = (g_m)^m$.

   b) Assume now that $f \not\equiv 0$ and f has analytic mth roots for all m. Show that $f(z) \neq 0$ for all $z \in D$.
   Hint: Look at multiplicities.

4. Suppose $f: B \to B$ is analytic, $|f(z)| \to 1$ as $|z| \to 1$, and f has exactly one zero in B (counting multiplicities). Show that f is one-to-one and onto.
   Hint: For a given $\lambda \in B$, show that f and $f - \lambda$ have the same number of zeros in B.

5. Let $\mathcal{F}$ denote the family of functions f analytic on B so that $f(0) = 0$ and

$$|f'(z)| \leq \frac{1}{1 - |z|}$$

   for all $z \in B$. Show that $\mathcal{F}$ is a normal family.

6. Let E be a set with no limit point in $\mathbf{C}$. Show that there exists a meromorphic function f on $\mathbf{C}$ so that f has a pole of order two at each point of E and f has no zero in $\mathbf{C}$.