Comprehensive Examination
Real Analysis
January 1998

A passing mark will be given if 5 problems are completely solved.

1. Suppose that $(X, \mathcal{A})$ is a measurable space. Show that if f and g are measurable real-valued functions on X, then $f+g$ is measurable.
2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable real-valued functions on X.
   (a) Define $(f_n)_{n=1}^{\infty}$ *converges in measure.*
   (b) Describe an example of a sequence which converges in measure but does not converge almost everywhere.
3. Let m be Lebesgue measure on R and fix a measurable set A_0 with $m(A_0) < \infty$. Define the measure ν on the Lebesgue measurable sets by $\nu(E) = m(A_0 \cap E)$. Find the Radon–Nikodym derivative of ν with respect to m. Justify your answer.
4. Let f be a function from $[0,1]$ into $\mathbf{R}$.
   (a) Define f *is absolutely continuous.*
   (b) Show that if f is a Lipschitz function on [0,1], it is absolutely continuous. (f is Lipschitz if there exists a constant K such that $|f(x) - f(y)| \leq K|x-y|$ for all x, y.)
   (c) Is every absolutely continuous function on [0,1] Lipschitz? Justify your answer.
5. Suppose that μ is a finite regular Borel measure, i.e., a Radon measure, on $[-1,1]$ such that for $n = 0, 1, 2, \ldots$, $\int x^n \, d\mu = 0$. Show that $\mu = 0$.
6. Let $C([0,1])$ be the space of continuous functions on $[0,1]$ with $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$.
   (a) State the Riesz Representation Theorem for $C([0,1])^*$?
   (b) Suppose that (f_n) is a sequence in $C([0,1])$ with $\|f_n\| \leq 1$ for all n such that $\lim_{n\to\infty} f_n(t) = 0$ for all t, show that (f_n) converges to 0 weakly.