Comprehensive Examination
Advanced Calculus
January 1999

A passing mark will be given if 5 problems are completely solved.

1. Determine whether each of the following statements is true or false. If it is true, prove it; if it is false, give a counterexample. In each case $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers.
   (a) The series $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} a_n^2$ converges.
   (b) The series $\sum_{n=1}^{\infty} a_n$ converges if $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

2. Suppose $0 < p < 1$. Prove that $t^p + 1 \geq (1+t)^p$ for all $t \geq 0$.

3. Let
$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$
   Prove or disprove f is differentiable at $(0,0)$.

4. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of continuous real-valued functions on a metric space (X,d).
   (a) Define $(f_n)_{n=1}^{\infty}$ *converges uniformly* to $f : X \to \mathbb{R}$.
   (b) Prove that if X is the real line and $(f_n)_{n=1}^{\infty}$ converges uniformly to f, then f is continuous.

5. Let X, Y and Z be metric spaces and assume that Y is compact. Suppose that $f : X \to Y$, $g : Y \to Z$ and that g is continuous and one-to-one. Define $h(x) = g(f(x))$ for all $x \in X$. Prove that f is uniformly continuous if h is uniformly continuous.

6. Define
$$f(x) = \begin{cases} \sin(\frac{1}{x}) & 0 < x \leq 1, \\ 0 & x = 0, \end{cases} \quad \text{and } f_n(x) = [f(x)]^n \qquad \text{for } n = 1, 2, \ldots.$$
   (a) For each $x \in [0,1]$ determine whether or not $\lim_{n\to\infty} f_n(x)$ exists and carefully justify your results.
   (b) Compute $\lim_{n\to\infty} \int_{\frac{1}{\pi}}^{1} f_n(x)\, dx$ and justify your computation.

7. Let $f : [a,b] \to \mathbb{R}^n$ for some $n > 1$.
   (a) Define f *is rectifiable.*
   (b) Prove f is not rectifiable if
$$f(x) = \begin{cases} (x, x\sin(\frac{1}{x})) & x \in (0,1], \\ (0,0) & x = 0. \end{cases}$$