MASTERS COMPREHENSIVE EXAM
ADVANCED LINEAR ALGEBRA
JANUARY 1999

(1) Consider the following matrices:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ --1 & 0 & 0 & 1 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Group these matrices into similarity classes. (Recall A_j is similar to A_k if and only if $A_j = QA_kQ^{-1}$ for some invertible matrix Q. Then for each A_j group together all A_k's similar to A_j.)

(b) Find the minimal polynomial of each A_j.

(2) What condition on the complex numbers α, β and γ is equivalent to the matrix

$$\begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

being diagonalizable (as a complex matrix)?

(3) Consider the matrix

$$A = \begin{pmatrix} 4 & 7 \\ -2 & -4 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find the eigenvalues of $A^{15} + 2A + I$. (Hint: You do not need to calculate powers of A.)

(4) (a) Is there a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that

$$\begin{aligned} T\big((1,3,0,1)\big) &= (1,0,0), \\ T\big((1,1,1,-1)\big) &= (1,1,0), \text{ and} \\ T\big((-1,3,-3,5)\big) &= (1,0,1)? \end{aligned}$$

Explain your answer.

(b) Is there a linear map $S : \mathbb{R}^4 \to \mathbb{R}^3$ such that

$$\begin{aligned} S\big((1,2,1,1)\big) &= (1,2,0), \\ S\big((1,0,1,0)\big) &= (1,1,0), \text{ and} \\ S\big((0,3,0,1)\big) &= (1,2,1)? \end{aligned}$$

Explain your answer.