Advanced Calculus Comprehensive Exam
January 2000

Four problems completely solved will constitute a pass.

(1) Suppose that (X, d) is a complete metric space and (x_n) is a sequence in X such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$. Prove that (x_n) converges.

(2) Consider $f_n(x) = nx \exp(-nx)$.

(a) Compute $\lim_{n\to\infty} f_n(x)$.

(b) Prove f_n converges uniformly on $[\delta, \infty)$ for every $\delta > 0$.

(c) Prove or disprove: f_n converges uniformly on $[0, \infty)$.

(3) Let a_n be a sequence of positive real numbers so that $\sum_{n=1}^{\infty} a_n = \infty$.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n} = \infty.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n^2 a_n} < \infty.$$

(4) Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$

wherever the series converges.

(a) Determine the domain for f.

(b) Determine the set of points where f is continuous.

(5) Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(u, v) = (uv, \frac{u^2 - v^2}{2}).$$

(a) Compute the Jacobian of T.

(b) Note that $T(0, 1) = (0, -\frac{1}{2})$. Do there exist neighborhoods A of $(0, 1)$ and B of $(0, -\frac{1}{2})$ such that there is a differentiable function $S : B \to A$ satisfying $S(T(a)) = a$ for all $a \in A$? Briefly explain your reasoning.

(c) Is T one–to–one?

(d) Sketch $\mathcal{R} = T([0,1] \times [0,1])$.

(e) Compute the integral

$$\int_{\mathcal{R}} y \, dx dy.$$