MASTER'S COMPREHENSIVE EXAMS

Advanced Linear Algebra

January 2000

1. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find a matrix Q so that $Q^{-1}AQ$ is a diagonal matrix.

(c) Find the minimal polynomial of A.

(d) Find A^n for every positive integer n.

(e) Give a geometric description of A.

2. Let V be an inner product space *i.e.* a real vector space with an inner product which we denote by (x, y). Recall that a linear transformation T of V is said to be *symmetric* or *self adjoint* if $(T(x), y) = (x, T(y))$ for all $x, y \in V$. Let T be a symmetric linear map. Prove the following two facts.

(a) $\ker(T)$ and $\operatorname{ran}(T)$ are orthogonal.

(b) $V = \ker(T) \oplus \operatorname{ran}(T)$.

3. Let $\mathbf{R^{n\times n}}$ be the vector space of all $n \times n$ matrices with real entries. For each **nilpotent** matrix $A \in \mathbf{R^{n\times n}}$ consider the dimension of $\operatorname{span}\{A^k : k = 0, 1, 2, \ldots\}$ (a subspace of $\mathbf{R^{n\times n}}$). What is the greatest this dimension can be? Explain your answer. (Here A^0 is the identity matrix.)

4. Let $p(x) = (x-2)^2(x-4)^3$. Determine all minimal polynomials of 5×5 matrices that have $p(x)$ as characteristic polynomial. Also find all Jordan canonical forms of such matrices.