Masters Algebra Examination
Oklahoma State University
Department of Mathematics
Winter, 2000

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Let $A = (a_{ij})$ be the $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } i+1, \\ 0 & \text{otherwise} \end{cases}$$

   Prove that A is invertible and find an explicit formula for the ij-entry b_{ij} of the inverse $A^{-1} = (b_{ij})$.

2. Let V be a finite-dimensional real vector space equipped with an inner product (u, v) on pairs of vectors. Let W be a subspace of V.
   (a) Define the orthogonal complement $W^\perp$ of W.
   (b) Prove that $V = W \oplus W^\perp$.

3. (a) Classify all groups of order 6.
   (b) What is the definition of the Class Equation for a group?
   (c) Write down all the possible Class Equations for groups of order 6. You should justify your answer.

4. If p and q are primes, show that there is no simple group of order pq.

5. Let R be an integral domain, and let F be the *field of fractions* or *quotient field* of R.
   (a) Explain how to construct the elements of F.
   (b) Define multiplication on F and show that it is well-defined.
   (c) Define addition on F and show that it is well-defined.
   (d) Show that every nonzero element of F has a multiplicative inverse.
   (e) Define an injective homomorphism of R into F.

6. Let F be a field. Let $R_1 = F[x]/(x^2 - 2)$ and $R_2 = F[x]/(x^2 - 3)$. Determine whether or not R_1 and R_2 are isomorphic rings in each of the cases $F = \mathbb{C}, \mathbb{R}, \mathbb{F}_2, \mathbb{F}_5$, and $\mathbb{F}_{11}$.