MASTER'S EXAM - TOPOLOGY - JANUARY 2000

1. Prove that every closed subspace of a compact topological space is compact.
2. Let $\mathcal{B}$ be a countable basis for the topology $\mathcal{T}$ on the set X. Let $\mathcal{C}$ be an arbitrary basis for $\mathcal{T}$. Prove that $\mathcal{C}$ has a countable subset $\mathcal{D}$ which is a basis for $\mathcal{T}$. (Hint: Suppose $\mathcal{B} = \{B_n\}_{n \in \mathbf{Z}_+}$. Consider the set of those pairs of indices (i, j) such that there exists a $C \in \mathcal{C}$ with $B_i \subseteq C \subseteq B_j$.)
3. Prove that every connected normal space having at least two points is uncountable. (Hint: Urysohn Lemma.)
4. Let (X, d) be a metric space. Let $x_0 \in X$. Let $U = \{x \in X \mid d(x, x_0) > 1\}$. Prove that U is open in X.
5. Let (X, d) be a compact metric space. Recall that if $\mathcal{A}$ is any covering of X by open sets, then there is a number $\lambda > 0$, called the Lebesgue number of the covering, such that if C is any set of diameter less than λ, then there exists an element A of $\mathcal{A}$ such that $C \subseteq A$. Use this fact to prove that if $f : X \to \mathbf{R}$ is continuous, then f is uniformly continuous.