MASTER'S COMPREHENSIVE EXAMS

Numerical Analysis

January 2000

INSTRUCTIONS: Calculator required. Show eight or more decimal figures. Three problems completely solved will be sufficient for a passing grade.

1. Consider the following data (at right).

x	$f(x)$
0.0	0.0
0.2	0.19867
0.4	0.38942
0.6	0.56464
0.8	0.71736
1.0	0.84147

a) Find the Newton form of the interpolating polynomial.

b) Use this form to approximate $f(0.5)$.

c) Given that the above data actually comes from $f(x) = \sin(x)$, obtain a bound on the error (without computing the actual value).

d) Compare with the actual error.

2. Prove the iteration (Schröder series)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2\, f''(x_n)}{2\, f'(x_n)^3}$$

has at least cubic order of convergence to a root of the equation $f(x) = 0$ when started in the neighborhood of a solution, under reasonable assumptions on the function $f(x)$ and its derivatives $f'(x)$, $f''(x)$. Be explicit about applicable definitions and required conditions.

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3.a) For $\int_a^b f(x)\,dx$ state formulas defining the Composite Trapezoidal Rule T_N, the Composite Midpoint Rule M_N, and the Composite Simpson Rule S_N on N subintervals of equal length.

b) Show the following formulas are correct (these are useful for efficient coding of the Romberg table):

$$\begin{aligned} T_{2N} &= \frac{1}{2}(T_N + M_N); \\ S_N &= \frac{4T_{2N} - T_N}{3} \\ &= \frac{T_N + 2\,M_N}{3}. \end{aligned}$$

4. Consider the matrix $A = \begin{bmatrix} 2.00 & -1.99 \\ -1.98 & 1.97 \end{bmatrix}$. Solving the equation $Ax = b$ with $b = \begin{bmatrix} 7.98 \\ -7.90 \end{bmatrix}$, the solution vector is x, while the right–hand side vector $\hat{b} = \begin{bmatrix} 7.9801 \\ -7.8999 \end{bmatrix}$ results in $\hat{x}$ as solution of $A\hat{x} = \hat{b}$.

a) Compute the solution vectors x, $\hat{x}$ as well as the error vector $e = \hat{x} - x$ and the residual vector $r = \hat{b} - b$.

b) Using the ∞–norm, $\|x\| = \max_{j=1..n} |x_j|$, determine numerically the norm–relative error, $e_{rel} = \|e\|/\|x\|$ as well as the norm–relative residual, $r_{rel} = \|r\|/\|b\|$, and decide whether the ratio e_{rel}/r_{rel} is maximal for this given matrix A.

HINT: You will need the condition number of the matrix.