Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
Winter, 2000

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. For a positive integer n, let S_n denote the group of permutations of n objects.
   (a) Prove that every automorphism of S_3 is of the form $\sigma \to \tau\sigma\tau^{-1}$ for some $\tau \in S_3$.
   (b) Find a sequence of subgroups
   $$1 = G_1 \subset G_2 \subset \cdots \subset G_n = S_4$$
   such that, for all $1 \leq j \leq n-1$, G_j is a normal subgroup of G_{j+1} and the quotient group G_{j+1}/G_j is abelian.

2. (a) What is the definition of the center of a group?
   (b) Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime dividing $|G|$. Prove that H is in the center of G.

3. Let R be an integral domain. Prove there is a field K and an embedding $\phi : R \hookrightarrow K$ with the property that, for any other embedding $\psi : R \hookrightarrow E$ into a field E, there is an embedding $\Psi : K \hookrightarrow E$ such that $\Psi \circ \phi = \psi$.

4. (a) What is the definition of a simple module over a ring?
   (b) Let $\phi : S \to S'$ be a homomorphism of simple modules over a ring R. Prove that either $\phi = 0$, or ϕ is an isomorphism.

5. Let t be transcendental over F and let $K = F(t)$. Let E be an intermediate field $F \subset E \subset K$ with $E \neq F$. Show that t is algebraic over E.

6. Let $K = \mathbb{Q}(i, \sqrt{2})$.
   (a) Prove that $K/\mathbb{Q}$ is a Galois extension, and determine its galois group.
   (b) Determine with proof which, if any, of the following fields is a Galois extension of $\mathbb{Q}$.
      i. $K(\sqrt{1+\sqrt{2}})$;
      ii. $K(\sqrt{i+\sqrt{2}})$;
      iii. $K(\sqrt{\sqrt{2}+\sqrt{-2}})$.