Comprehensive Exam
Real Analysis
January 2000

A passing mark will be given if 5 problems are completely solved.

1. If $\mathcal{X}$ is a Banach space and x is any element of $\mathcal{X}$ then there is a functional $x^* \in \mathcal{X}^*$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$. Show that this happens in both the Banach space $C[0,1]$ and in the Banach space $L^1[0,1]$. More specifically do the following:
(a) If $f \in C[0,1]$ describe a functional $x^* \in C[0,1]^*$ so that $x^*(f) = ||f||_\infty$.
(b) If $f \in L^1[0,1]$ describe a functional $x^* \in L^1[0,1]^*$ so that $x^*(f) = ||f||_1$.
2. Let (A_n) be a sequence of sets in an algebra $\mathcal{A}$ of sets. Show that there is a disjoint sequence (B_n) of sets in $\mathcal{A}$ such that, for each k,
$$\cup_{n=1}^k A_n = \cup_{n=1}^k B_k.$$
3. Consider the following 3 measures, μ_0, μ_1, and μ_2, defined on the measurable space $([0,1], \mathcal{M})$ of Lebesgue measureable sets in $[0,1]$.
μ_0 is Lebesgue measure.
μ_1 is the counting measure defined on $\mathcal{M}$.
μ_2 is defined for $E \in \mathcal{M}$ by $\mu_2(E) = \int_E x^2\, dx$.
Indicate for exactly which ordered pairs of integers i and j it is true that $\mu_i \ll \mu_j$.
4. Let $(f_n)_{n=1}^\infty$ be a sequence of measurable real-valued functions on $[0,1]$ such that $f_n \in L_2[0,1]$ for all $n \in \mathbf{N}$ and $\sum_{n=1}^\infty \int |f_n| < \infty$.
(a) Show that $f = \lim_{N\to\infty} \sum_{n=1}^N f_n$ is defined and finite a.e.
(b) Show that f, defined as in the previous part, need not be in $L_2[0,1]$.
5. Let E be a Borel measurable subset of $[0,1]$ and let χ represent the characteristic or indicator function of E, i.e., $\chi(x) = 1$ if $x \in E$ and $\chi(x) = 0$ otherwise. Show that for any $\epsilon > 0$, there is a continuous function $f \in C[0,1]$ such that
$$\int_0^1 |f - \chi| dx < \epsilon.$$
6. Let $K = [0,1] \times [0,1]$ be the unit square. Let $f \in C(K)$ and $\epsilon > 0$ be given. Show that there is a function $g \in C(K)$ such that $||f - g||_\infty < \epsilon$ and g has the the form
$$g(x,y) = \sum_{i=1}^n a_i f_i(x) g_i(y)$$
where f_i and g_i are in $C[0,1]$.