1. Find all integers of the form $n^4 + 4$ which are prime numbers.

   Hint: $n^4 + ? + 4$ is a perfect square.

   Solution *(Note: Although not stated, it has been tactically assumed that n is an integer. After all, if n is allowed to be any real numbers, then any prime $p \geq 5$ can be expressed as $n^4 + 4$.)*

   Notice that

$$\begin{aligned} n^4 + 4 &= (n^4 + 4n^2 + 4) - 4n^2 \\ &= (n^2 + 2)^2 - (2|n|)^2 \\ &= (n^2 + 2 - 2|n|)(n^2 + 2 + 2|n|) \end{aligned}$$

   Clearly for all $|n| > 1$, both $n^2 + 2 - 2|n|$ and $n^2 + 2 + 2|n|$ are integers greater than 1. In other words, when $|n| > 1$, the number $n^4 + 4$ can not be a prime number.

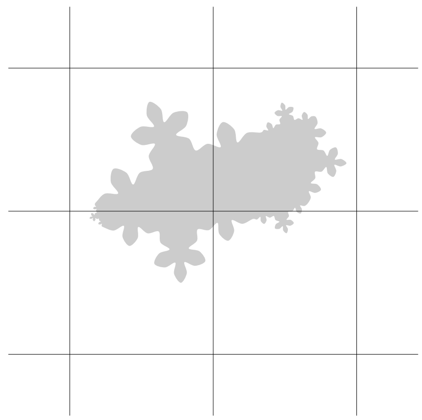
   The only choices left are $n = -1, 0, 1$. We have

$$\begin{aligned} (-1)^4 + 4 &= 5 \quad \text{is a prime number,} \\ 0^4 + 4 &= 4 \quad \text{is not a prime number,} \\ 1^4 + 4 &= 5 \quad \text{is a prime number.} \end{aligned}$$

   Hence, 5 is the only integer of the form $n^4 + 4$ which is a prime number.

2. Suppose that S is a subset of the plane, like the shaded area in the figure below, and the area of S is greater than 1. Show that there exist two points $p = (x_1, y_1)$ and $q = (x_2, y_3)$ in S such that $x_1 - x_2$ and $y_1 - y_2$ are both integers.

   Hint: Cut the plane into a grid of squares of side length 1, as in the figure. Now imagine stacking the pieces on top of each other...



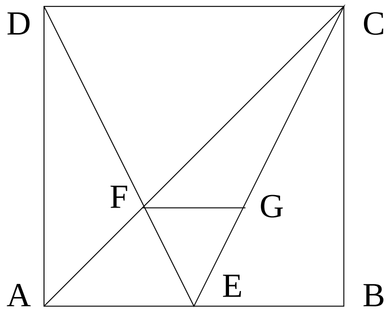
Solution *(There is a typo in the problem. Point q should have coordinate (x_2, y_2) instead of (x_2, y_3).)*

In the grid of squares, any two points, which belong to different squares but are located at the same geometric position in their own squares, must have integer differences in their x- and y-coordinates.

When stacking the 1×1 squares on top of each other, if the shaded area (subset S) overlaps, the overlapping part must be points located at the same geometric position in different squares. According to the previous analysis, the differences in their x- and y-coordinates are both integers. Now the only thing left is to show that the shaded area (subset S) must overlap in the stacking process.

This is true because the total area of S is greater than 1. If we try to cut, stack and fit it into a 1×1 square whose area is only 1, there must be some overlap.

3. The square $\Box ABCD$ has sides of length 2. Point E lies on the center of edge AB. Point F is the intersection of lines AC and DE. Line FG is parallel to line AB. Find the area of triangle ΔEFG.



Solution There are many different methods for solving this problem. Here we only give one of them.

Clearly, AE has length 1. Triangle ΔACE has base 1 and height 2, hence its area is 1.

Triangle ΔAEF and ΔCDF are similar triangles. Because AE has half the length of CD, we know that AF must have half the length of CF. Also because FG is parallel to AE, consequently EG must have half the length of CG.

Now consider ΔACE with AC as its base. By comparing the ratio of AF and CF, we conclude that the area of ΔCFE is 2/3 of the area of ΔACE. Hence the area of ΔCFE is 2/3.

Finally, consider ΔCFE with EC as its base. By comparing the ratio of EG and CG, we conclude that the area of ΔEFG is 1/3 of the area of ΔCFE. Therefore, the area of ΔEFG is $(1/3) \times (2/3) = 2/9$.

4. Let

$$x = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}}$$

Show that $1 < x < 2$.

Solution Notice that the equation may be rewritten as

$$x = \sqrt[3]{1 + x}.$$

or $x^3 = 1 + x$. This implies that $x^3 - x = 1$.

Now, since $x = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}}$, clearly $x > \sqrt[3]{1} = 1$.

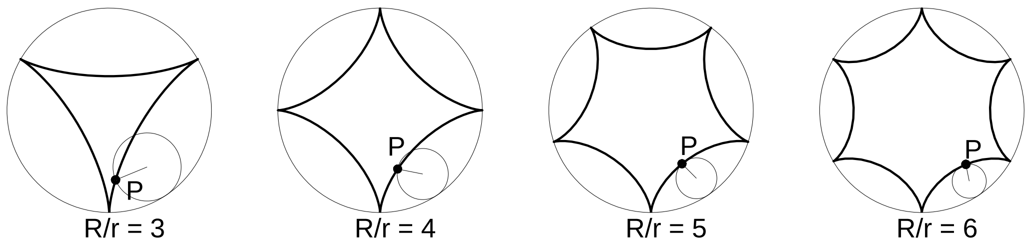
Suppose $x \geq 2$, then $x^3 - x = x(x-1)(x+1) \geq 2 \cdot 1 \cdot 3 = 6$. So it is not possible that $x^3 - x = 1$. Hence $x \geq 2$ is not possible.

Putting both together, we have $1 < x < 2$.

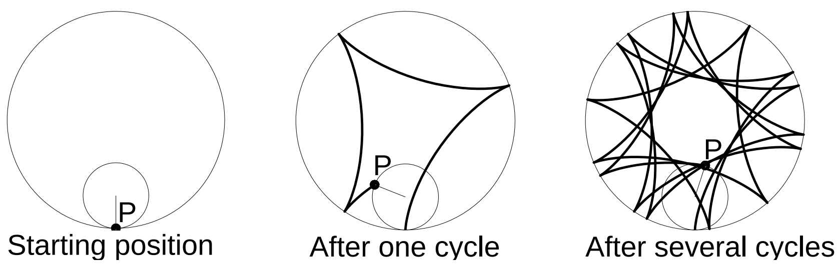
(To be mathematically strict, one should prove that the limit of the infinite sequence exists. However, this is beyond the scope of high school mathematics. So here we simply assume the limit exists.)

5. When a small circle of radius r rolls around (no sliding) the inside of a large circle of radius R, the trace of a point on the small circle is called a *hypocycloid*. We assume that both R and r are integers.

   When the ratio R/r is an integer, P will return to its starting position after one full cycle of rolling around the large circle. In the diagrams below, we illustrate the trace of P with $R/r = 3$, 4, 5, 6, in thick dark curves.



   When R/r is not an integer, P will not return to its starting position after one full rolling cycle. However, we can keep rotating the small circle and the trace of P is illustrated in below.



   In this case, will P ever be able to go back to its starting position? If your answer is no, explain why. If your answer is yes, calculate the total number of full cycles (around the large circle) needed for P to go back to its starting position.

   Solution Yes, P will go back to its original position after n full cycles around the large circle if the total distance travelled, $n(2\pi R)$, is a multiple of the inner circle's circumference $2\pi r$. In other words, P will go back to its original position after n full cycles if

$$\frac{n(2\pi R)}{2\pi r} = n\frac{R}{r}$$

   is an integer. Since R and r are integers, let $d = gcd(R, r)$ be the greatest common divisor of R and r, then

$$n\frac{R}{r} = n\frac{R/d}{r/d}$$

   is an integer if n is a multiple of r/d.