Heegaard Floer Homology Exercise Set #4

Exercise 1: Show that

$$\widehat{HF}(\overset{n}{\#}(S^1 \times S^2)) \cong \bigoplus_{k=0}^{n} A_k$$

where

$$A_k := \mathbb{F}^{\binom{n}{k}}$$

with the following relative grading. For all pairs $0 \leq k, l \leq n$, given $x \in A_k$ and $y \in A_l$,

$$gr(x, y) = k - l.$$

Exercise 2:

1. Show that if (Y_1, Y_2, Y_3) is a triad than so are (Y_2, Y_3, Y_1) and (Y_3, Y_1, Y_2).

2. Let $|H_1(Y; \mathbf{Z})|$ be the cardinality of $H_1(Y; \mathbf{Z})$ if it's finite and 0 if $H_1(Y; \mathbf{Z})$ is infinite. Show that if (Y_1, Y_2, Y_3) is a triad then in some cyclic order

$$|H_1(Y_1; \mathbf{Z})| + |H_1(Y_2; \mathbf{Z})| = |H_1(Y_3; \mathbf{Z})|.$$

   Show that in that order, if Y_1 and Y_2 are L-spaces that so is Y_3. (Hint: Given a rational homology sphere Y for each $s \in Spin^c(Y)$, $\widehat{HF}(Y, s)$ is nontrivial.)

3. Let K be a null-homologous knot in a 3-manifold Y. Show that there is a unique longitude λ null-homologous in $Y - \nu(K)$ where $\nu(K)$ is a neighborhood of K. This longitude is called the *Seifert longitude* of K in Y.

4. Let K be a knot is S^3, and let λ be the Seifert longitude of K in S^3. Let p and q be co-prime integers (or (p, q) is $(1, 0)$ or $(0, 1)$). Denote surgery of S^3 along K with framing $q\lambda + p\mu$ by $S^3_{\frac{p}{q}}(K)$. Let p_1, p_2, q_1, and q_2 be integers such that $p_1 q_2 - p_2 q_1 = 1$. Let $p_3 = p_1 + p_2$ and $q_3 = q_1 + q_2$. Show that $(S^3_{\frac{p_1}{q_1}}(K), S^3_{\frac{p_2}{q_2}}(K), S^3_{\frac{p_3}{q_3}}(K))$ is a triad. In particular, for any integer n, $(S^3, S^3_n(K), S^3_{n+1}(K))$ is a triad.

5. Show that for a knot K in S^3, if for some rational $r > 0$, $S^3_r(K)$ is an L-space then $S^3_n(K)$ is an L-space for all integers $n > r$.

Exercise 3: Let L be a link is S^3. Denote the unique branched double cover of S^3 branched along L by $\Sigma(L)$. Select a crossing in a diagram of L and let L_0 and L_1 be the links represented by resolving the crossings as in Figure 1. Show that $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$ is a triad.

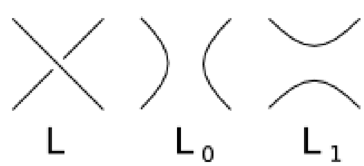

Figure 1: Resolutions of a crossing of L