Exercise Set #3

Exercise 1: Suppose Q is an integral form. Show that the following are equivalent:

(i) Q is even.

(ii) Every matrix representation of Q has diagonal with all even entries.

(iii) At least one matrix representation of Q has diagonal with all even entries.

Exercise 2: Compute $\mathsf{sign}(E_8)$.

$$E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

Exercise 3: Prove that any closed oriented simply-connected 4-manifold with even intersection form and vanishing signature is homeomorphic to S^4 or a connected sum of some number of $S^2 \times S^2$.

Exercise 4:

a) Given a bilinear integral form $Q : L \times L \to \mathbb{Z}$, define $F : L \to L^*$ by $F(x) = F_x$ where $F_x(y) = Q(x, y)$. Show that Q is unimodular if and only if F is an isomorphism.

b) Show that if M is a closed simply-connected orientable 4-manifold, then the intersection form Q_M is unimodular. *(Hint: Use Poincaré duality.)*

Exercise 5: Let U_1 and U_2 be disjoint (not necessarily unlinked) framed unknots in the boundary of a 0-handle, B, with framings n_1 and n_2. Let M be the 4-manifold obtained by attaching 2-handles to B along U_1 and U_2. For $i = 1, 2$, let F_i be the sphere obtained by pushing the interior of a disk in ∂B bound by U_i into the interior of B and capping of the disk with the core of the attached 2-handle. Compute $F_1 \cdot F_2$ and $F_1 \cdot F_1$.

Exercise 6: Let M be a 4-dimensional manifold constructed by attaching 1- and 2-handles to a 0-handle.

a) Let M' be the manifold obtained by attaching a zero framed 2-handle to the belt sphere of one the 2-handles of M. Show that $\pi_1(M) \cong \pi_1(M')$.

b) Find a handle decomposition of the *double* of M which is $DM := M \cup_{id_{\partial M}} \overline{M}$.

c) Let G be a finitely presented group. Find a closed orientable 4-manifold M with $\pi_1(M) \cong G$.